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A KEY

TO

ELEMENTARY TRIGONOMETRY

BY

J. HAMBLIN SMITH, M.A.

OF GONVILLE AND CAIUS COLLEGE,
AND LATE LECTURER AT ST. PETER'S COLLEGE, CAMBRIDGE

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PREFACE.

I HAVE to acknowledge most gratefully the assistance rendered me in the preparation of this book by Mr. T. H. Gascoigne, son of the Rev. T. Gascoigne, of Spondon House School, Derby. For the solutions of a few of the Problems I am indebted to Mr. Gaskin's *Trigonometrical Examples*, and to Mr. Hymers' *Trigonometry*. I shall be glad to receive corrections of errors that may be discovered in my work.

CAMBRIDGE, *October* 1876.

ELEMENTARY TRIGONOMETRY.

KEY.

EXAMPLES—I. (pp. 1, 2).

(1) 4 feet 6 inches $=54$ inches ; $\therefore$ number is 54.

(2) 15 feet 2 inches $=182$ inches ; $\therefore$ number is $182\div 7$, or, 26.

(3) Unit of square measurement is $(192\div 12)$ square inches, or, 16 square inches ; $\therefore$ unit of linear measurement is $\sqrt{16}$ inches, or, 4 inches.

(4) Unit of square measurement is $(1000\div 40)$ square inches, or, 25 square inches; $\therefore$ unit of linear measurement is $\sqrt{25}$ inches, or, 5 inches.

(5) Unit of cubic measurement is $(216\div 8)$ cubic inches, or, 27 cubic inches ; $\therefore$ unit of linear measurement is $\sqrt[3]{27}$ inches, or, 3 inches.

(6) Unit of cubic measurement is $(2000\div 16)$ cubic inches, or, 125 cubic inches; $\therefore$ unit of linear measurement is $\sqrt[3]{125}$ inches, or, 5 inches.

(7) Measure of 1 yard is $\dfrac{1}{a}$;

$\therefore$ measure of 1 foot is $\dfrac{1}{3a}$;

$\therefore$ measure of b feet is $\dfrac{b}{3a}$.

(8) Length of line is (5×6) inches, or, 30 inches; $\therefore$ second unit is $(30 \div 4)$ inches, or, $7\frac{1}{2}$ inches.

(9) Length of line is 1 yard, or, 36 inches;
$\therefore$ second unit is $(36 \div 36)$ inches, or, 1 inch;
and third unit is $(36 \div 12)$ inches, or, 3 inches.

(10) The ratio is $3\frac{1}{4} : 3\frac{1}{2} \times 36$, or, $3\frac{1}{4} : 126$, or, $13 : 126 \times 4$, or, $13 : 504$.

(11) The measure of 1 yard is $\frac{c}{m}$;

$\therefore$ the measure of 1 foot is $\frac{c}{3m}$;

$\therefore$ the measure of n feet is $\frac{nc}{3m}$.

Examples—II. (p. 3).

(1) Length of other side $= \sqrt{(51)^2 - (24)^2}$ yards $= \sqrt{2601 - 576}$ yards $= \sqrt{2025}$ yards $= 45$ yards.

(2) Length of hypothenuse $= \sqrt{8^2 + 6^2}$ feet $= \sqrt{64 + 36}$ feet $= \sqrt{100}$ feet $= 10$ feet.

(3) Diagonal $= \sqrt{\{(225)^2 + (120)^2\}}$ yards $= \sqrt{65025}$ yards $= 255$ yards.

(4) Diagonal $= \sqrt{\{(300)^2 + (200)^2\}}$ yards $= \sqrt{130000}$ yards $= 360\cdot5 \ldots$ yards.

(5) Length $= \dfrac{2\frac{1}{2} \times 4840}{88}$ yards $= (2\cdot5 \times 55)$ yards $= 137\cdot5$ yards;

diagonal $= \sqrt{\{(137\cdot5)^2 + (88)^2\}}$ yards $= \sqrt{26650\cdot25}$ yards $= 163\cdot25$ yards, nearly.

(6) Let $x + y$, x, $x - y$ be the length of the sides in feet.

$$\text{Then } (x+y)^2 = x^2 + (x-y)^2,$$
$$\text{or, } x^2 + 2xy + y^2 = x^2 + x^2 - 2xy + y^2,$$
$$\text{or, } 4xy = x^2, \text{ and } \therefore x = 4y.$$

Hence $x + y = 5y$,
and $5y = 20$ feet, and $\therefore y = 4$ feet.
Hence the other sides are 16 feet and 12 feet.

(7) Proceeding as in Example (6), we get

$$x+y=5y\,;\ x=4y\,;\ x-y=3y.$$

Hence the sides are as $3y:4y:5y$, that is, as $3:4:5$.

(8) Let $AB=36$ feet, and $DE=27$ feet;

CA, CE being the two positions of the ladder.

Then since ACE is a right angle, $\angle\, ACB, ECD$, are together equal to a right angle.

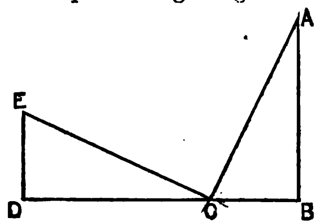

FIG. 1.

But $\angle^s ACB, CAB$ are together equal to a right angle,

and $\therefore\ \angle\, ECD=\angle\, CAB.$

Hence in $\triangle$s ABC, CDE.

right $\angle\, ABC=$right $\angle\, CDE$, and $\angle\, CAB=\angle\, ECD$, and $AC=CE$;

$\therefore\ BC=ED$, and $AB=CD$; (EUCLID, I. XXVI.)

$\therefore$ width of street $=(27+36)$ feet $=63$ feet,

and length of ladder $=\sqrt{(27)^2+(36)^2}$ feet $=\sqrt{2025}$ feet $=45$ feet.

(9) Let $x=$length of each of the equal sides in feet.

Then $x^2+x^2=(12)^2$, or, $2x^2=144$, or, $x^2=72$, or, $x=6\sqrt{2}.$

(10) Diagonal$=\sqrt{25+25}$ inches $=\sqrt{50}$ inches $=5\sqrt{2}$ inches.

(11) Each side of square $=\sqrt{390625}$ feet $=625$ feet;

$\therefore$ diagonal$=\sqrt{\{(625)^2+(625)^2\}}$ feet$=\sqrt{\{2\,.\,(625)^2\}}$ feet$=625\sqrt{2}$. feet.

(12) AD bisects BC.

$\therefore\ AB=2.BD.$

Let $x=$measure of length of AD.

Then $x^2=(13)^2-\left(\frac{13}{2}\right)^2$

$=(13)^2\{1-\frac{1}{4}\}=(13)^2\cdot\frac{3}{4}.$

$\therefore\ x=\frac{13\sqrt{3}}{2}.$

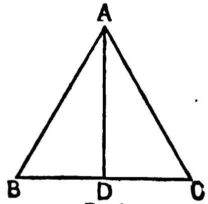

FIG 2.

(13) Taking the diagram in Example 12, let measure of AB be x.

Then $x^2 = \dfrac{x^2}{4} + (15)^2$;

$\therefore\ 3x^2 = 4 \times (15)^2$, or, $x^2 = \dfrac{4 \times (15)^2}{3} = \dfrac{4 \times (15)^2 \times 3}{9}$;

$\therefore\ x = \dfrac{2 \times 15\sqrt{3}}{3} = 10\sqrt{3}$.

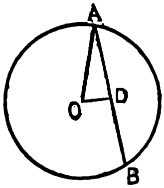

Fig. 3.

(14) OD, a perpendicular from the centre on the chord AB, bisects AB.

Let x = measure of OD in inches.

Then $x^2 = (OA)^2 - (AD)^2$
$= (37)^2 - (35)^2 = 144$;

$\therefore$ distance $= \sqrt{144}$ inches $= 12$ inches.

(15) Taking the diagram of Example (14).
Let measure of AD in inches be x.

Then $x^2 = (181)^2 - (180)^2 = 361$;

$\therefore\ x = 19$, and $\therefore\ AB = (2 \times 19)$ inches $= 38$ inches.

(16) Taking the diagram of Example (14).

Let measure of AO in feet be x.

Then $x^2 = (308)^2 + (75)^2 = 100489$;

$\therefore\ x = 317$, and $\therefore$ diameter $= (2 \times 317)$ feet $= 634$ feet.

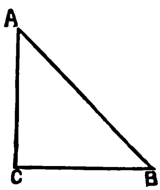

Fig. 4.

(17) $(AC)^2 + (BC)^2 = (AB)^2$.

$\therefore\ 2(AC)^2 = (AB)^2$;

$\therefore\ \dfrac{(AC)^2}{(AB)^2} = \frac{1}{2}$;

$\therefore\ \dfrac{AC}{AB} = \dfrac{1}{\sqrt{2}}$.

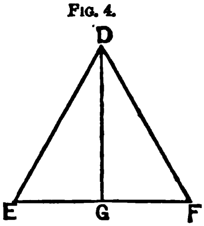

Fig. 5.

(18) Let x be the measure of EG.

Then $2x$ is the measure of ED;

and measure of $DG = \sqrt{(4x^2 - x^2)} = \sqrt{3}\,.\,x$;

$\therefore\ EG : ED : DG = x : 2x : \sqrt{3}\,.\,x$
$= 1 : 2 : \sqrt{3}$.

EXAMPLES—III. (p. 9).

(1) Circumference $=\dfrac{22\times 5}{7}$ feet $=\dfrac{110}{7}$ feet $=15\frac{5}{7}$ feet.

(2) Radius $=\dfrac{7\times 542\cdot 5}{44}$ feet $=\dfrac{3797\cdot 5}{44}$ feet $=86\cdot 306\dot{8}\dot{1}$ feet.

(3) Train goes in a second $\dfrac{22\times 12}{7}$ feet.

Rate in miles per hour $=\dfrac{22\times 12\times 60\times 60}{7\times 3\times 1760}=\dfrac{180}{7}=25\cdot\dot{7}1428\dot{5}$.

(4) Diameter in miles $=\dfrac{7\times 25000}{22}=7954\frac{6}{11}$.

(5) Circumference in miles $=\dfrac{22\times 883220}{7}=2775834\frac{2}{7}$.

(6) Radius in miles $=\dfrac{7\times 6850}{44}=\dfrac{23975}{22}=1089\frac{17}{22}$.

(7) Circumference in feet $=\dfrac{22\times 12\frac{1}{2}\times 2}{7}=\dfrac{22\times 25}{7}$;

$\therefore$ $\frac{1}{12}$ of circumference $=\dfrac{22\times 25}{12\times 7}$ feet $=6$ feet $6\frac{4}{7}$ inches.

(8) Circumference in feet $=\dfrac{22\times 21}{7}$;

$\therefore$ $\frac{5}{7}$ of circumference $=\dfrac{22\times 21\times 5}{7\times 7}$ feet $=47\frac{1}{7}$ feet.

(9) If x be the side of the square,

$(\text{diameter})^2=2x^2$;

$\therefore$ $x=\dfrac{\text{diameter}}{\sqrt{2}}=\dfrac{7\times 150}{22\times\sqrt{2}}$ feet $=\dfrac{7\times 150\times\sqrt{2}}{22\times\sqrt{2}\times\sqrt{2}}$ feet $=\dfrac{525\sqrt{2}}{22}$ feet.

(10) $x=\dfrac{\text{diameter}}{\sqrt{2}}=\dfrac{7\times 200}{22\times\sqrt{2}}$ feet $=\dfrac{7\times 200\times\sqrt{2}}{22\times 2}$ feet $=\dfrac{350\sqrt{2}}{11}$ feet.

(11) Point goes in a minute $\dfrac{22\times 12\times 30}{7}$ feet.

Rate in miles per hour $=\dfrac{22\times 12\times 30\times 60}{7\times 1760\times 3}=\dfrac{90}{7}=12\frac{6}{7}$.

(12) End goes in a minute $\dfrac{22 \times 2 \times 15 \times 21}{7}$ feet.

Rate in miles per hour $= \dfrac{22 \times 2 \times 15 \times 21 \times 60}{7 \times 3 \times 1760} = \dfrac{45}{2} = 22\frac{1}{2}$.

EXAMPLES—IV. (p. 12).

(1)
$$\begin{array}{r|l} 60 & 5\cdot \\ 60 & 16\cdot08\dot{3} \\ & \cdot2680\dot{5} \end{array}$$

$\therefore\ 24^\circ.\ 16'.\ 5'' = 24^\circ\cdot2680\dot{5}$

(2)
$$\begin{array}{r|l} 60 & 43\cdot \\ 60 & 2\cdot71\dot{6} \\ & \cdot0452\dot{7} \end{array}$$

$\therefore\ 37^\circ.\ 2'.\ 43'' = 37^\circ\cdot0452\dot{7}$

(3)
$$\begin{array}{r|l} 60 & 14\cdot \\ 60 & 0\cdot2\dot{3} \\ & \cdot003\dot{8} \end{array}$$

$\therefore\ 175^\circ.\ 0'.\ 14'' = 175\cdot003\dot{8}$.

(4)
$$\begin{array}{r|l} 60 & 28\cdot \\ 60 & 5\cdot4\dot{6} \\ & \cdot09\dot{1} \end{array}$$

$\therefore\ 5'.\ 28'' = \cdot09\dot{1}^\circ$.

(5)
$$\begin{array}{r|l} 60 & 4\cdot \\ & \cdot0\dot{6} \end{array}$$

$\therefore\ 375^\circ.\ 4' = 375^\circ\cdot0\dot{6}$.

(6)
$$\begin{array}{r|l} 60 & 4\cdot \\ 60 & 12\cdot0\dot{6} \\ & \cdot20\dot{1} \end{array}$$

$\therefore\ 78^\circ.\ 12'.\ 4'' = 78^\circ\cdot20\dot{1}$.

EXAMPLES—V. (p. 13).

(1) $25^g.\ 14'.\ 25'' = 25^g\cdot1425$.

(2) $38^g.\ 4'.\ 15'' = 38^g\cdot0415$.

(3) $214^g.\ 3'.\ 7'' = 214^g\cdot0307$.

(4) $15'.\ 7''\cdot45 = \cdot150745^g$.

(5) $425^g.\ 13'.\ 5''\cdot54 = 425^g\cdot130554$.

(6) $2^g.\ 2'.\ 2''\cdot22 = 2^g\cdot020222$.

EXAMPLES—VI. (p. 19).

(1) $27^\circ.\ 15'.\ 46'' = 27^\circ\cdot262\dot{7}$

$$\begin{array}{r} 10 \\ \hline |\ 272\cdot62\dot{7} \\ \hline 30\cdot291975\ \ldots \end{array}$$

$\therefore\ 27^\circ.\ 15'.\ 46'' = 30^g.\ 29'.\ 19''\cdot75\ \ldots$

(2) $157^\circ.4'.9''=157^\circ\cdot0691\dot{6}$

$$\begin{array}{r|l} & 10 \\ \hline 9 & 1570\cdot691\dot{6} \\ \hline & 174\cdot5212\dot{9}6\dot{2} \end{array}$$

$\therefore\ 157^\circ.\ 4'.\ 9''=174^g.\ 52'.\ 12''\cdot\dot{9}6\dot{2}.$

(3) $24'.\ 18''=0^\circ\cdot405$

$$\begin{array}{r|l} & 10 \\ \hline 9 & 4\cdot05 \\ \hline & \cdot45 \end{array}$$

$\therefore\ 24'.\ 18''=45'.$

(4) $19^\circ.0'.18''=19^\circ\cdot005$

$$\begin{array}{r|l} & 10 \\ \hline 9 & 190\cdot05 \\ \hline & 21\cdot1166\dot{6} \end{array}$$

$\therefore\ 19^\circ.\ 0'.\ 18''=21^g.\ 11'.\ 66''\cdot\dot{6}.$

(5) $143^\circ.9'=143^\circ\cdot15$

$$\begin{array}{r|l} & 10 \\ \hline 9 & 1431\cdot5 \\ \hline & 159\cdot0555\dot{5} \end{array}$$

$\therefore\ 143^\circ.\ 9'=159^g.\ 5'.\ 55''\cdot\dot{5}.$

(6) $28^\circ=28^\circ$

$$\begin{array}{r|l} & 10 \\ \hline 9 & 280 \\ \hline & 31\cdot1111\dot{1} \end{array}$$

$\therefore\ 28^\circ=31^g.\ 11'.\ 11''\cdot\dot{1}.$

(7) $10^\circ.\ 25'.\ 48''=10^\circ\cdot43$

$$\begin{array}{r|l} & 10 \\ \hline 9 & 104\cdot3 \\ \hline & 11\cdot5888\dot{8} \end{array}$$

$\therefore\ 10^\circ.\ 25'.\ 48''=11^g.\ 58'.\ 88''\cdot\dot{8}.$

(8) $27^\circ.\,38'.\,12''=27^\circ{\cdot}63\dot{6}$

$$\begin{array}{r} 10 \\ 9\;\underline{\big|\;276{\cdot}3\dot{6}} \\ 30{\cdot}7\dot{0}7\dot{4} \end{array}$$

$\therefore\ 27^\circ.\,38'.\,12''=30^g.\,70^{\backprime}.\,74^{\backprime\backprime}{\cdot}\dot{0}7\dot{4}.$

(9) $300^\circ.\,15'.\,58''=300^\circ{\cdot}266\dot{1}$

$$\begin{array}{r} 10 \\ 9\;\underline{\big|\;3002{\cdot}66\dot{1}} \\ 333{\cdot}6290\dot{1}23456789\dot{0} \end{array}$$

$\therefore\ 300^\circ.\,15'.\,58''=333^g.\,62^{\backprime}.\,90^{\backprime\backprime}{\cdot}\dot{1}23456789\dot{0}.$

(10) $422^\circ.\,7'.\,22''=422^\circ{\cdot}122\dot{7}$

$$\begin{array}{r} 10 \\ 9\;\underline{\big|\;4221{\cdot}22\dot{7}} \\ 469{\cdot}0253\dot{0}8641975\dot{3} \end{array}$$

$\therefore\ 422^\circ.\,7'.\,22''=469^g.\,2^{\backprime}.\,53^{\backprime\backprime}{\cdot}\dot{0}8641975\dot{3}.$

Examples—VII. (p. 20).

(1) $19^g.\,45^{\backprime}.\,95^{\backprime\backprime}=19^g{\cdot}4595$

$$\begin{array}{lr} & 9 \\ & 10\;\underline{\big|\;175{\cdot}1355} \\ \text{degrees} & 17{\cdot}51355 \\ & \underline{60} \\ \text{minutes} & 30{\cdot}81300 \\ & \underline{60} \\ \text{seconds} & 48{\cdot}78000 \end{array}$$

$\therefore\ 19^g.\,45^{\backprime}.\,95^{\backprime\backprime}=17^\circ.\,30'.\,48''{\cdot}78.$

(2) $124^g.\,5^{\backprime}.\,8^{\backprime\backprime}=124^g{\cdot}0508$

$$\begin{array}{lr} & 9 \\ & 10\;\underline{\big|\;1116{\cdot}4572} \\ \text{degrees} & 111{\cdot}64572 \\ & \underline{60} \\ \text{minutes} & 38{\cdot}74320 \\ & \underline{60} \\ \text{seconds} & 44{\cdot}59200 \end{array}$$

$\therefore\ 124^g.\,5^{\backprime}.\,8^{\backprime\backprime}=111^\circ.\,38'.\,44''{\cdot}592.$

(3) $29^g.75' = 29^g \cdot 75$

$$\begin{array}{lr}
 & 9 \\
10 & \underline{|\ 267 \cdot 75} \\
\text{degrees} & 26 \cdot 775 \\
 & \underline{60} \\
\text{minutes} & 46 \cdot 500 \\
 & \underline{60} \\
\text{seconds} & 30 \cdot 000
\end{array}$$

$\therefore\ 29^g.75' = 26°.46'.30''$.

(4) $15^g.0'.15'' = 15^g \cdot 0015$

$$\begin{array}{lr}
 & 9 \\
10 & \underline{|\ 135 \cdot 0135} \\
\text{degrees} & 13 \cdot 50135 \\
 & \underline{60} \\
\text{minutes} & 30 \cdot 08100 \\
 & \underline{60} \\
\text{seconds} & 4 \cdot 86000
\end{array}$$

$\therefore\ 15^g.0'.15'' = 13°.30'.4'' \cdot 86$.

(5) $154^g.7'.24'' = 154^g \cdot 0724$

$$\begin{array}{lr}
 & 9 \\
10 & \underline{|\ 1386 \cdot 6516} \\
\text{degrees} & 138 \cdot 66516 \\
 & \underline{60} \\
\text{minutes} & 39 \cdot 90960 \\
 & \underline{60} \\
\text{seconds} & 54 \cdot 57600
\end{array}$$

$\therefore\ 154^g.7'.24'' = 138°.39'.54'' \cdot 576$.

(6) $43^g = 43^g$

$$\begin{array}{lr}
 & 9 \\
10 & \underline{|\ 387} \\
\text{degrees} & 38 \cdot 7 \\
 & \underline{60} \\
\text{minutes} & 42 \cdot 0
\end{array}$$

$\therefore\ 43^g = 38°.42'$.

(7) $38^{g}.\,71'.\,20''{\cdot}3 = 38^{g}{\cdot}71203$

$$\begin{array}{lr} & 9 \\ 10 \,| & 348{\cdot}40827 \\ \text{degrees} & 34{\cdot}840827 \\ & 60 \\ \text{minutes} & 50{\cdot}449620 \\ & 60 \\ \text{seconds} & 26{\cdot}977200 \end{array}$$

$\therefore\ 38^{g}.\,71'.\,20''{\cdot}3 = 34^{\circ}.\,50'.\,26''{\cdot}9772.$

(8) $50^{g}.\,76'.\,94''{\cdot}3 = 50^{g}{\cdot}76943$

$$\begin{array}{lr} & 9 \\ 10 \,| & 456{\cdot}92487 \\ \text{degrees} & 45{\cdot}692487 \\ & 60 \\ \text{minutes} & 41{\cdot}549220 \\ & 60 \\ \text{seconds} & 32{\cdot}953200 \end{array}$$

$\therefore\ 50^{g}.\,76'.\,94''{\cdot}3 = 45^{\circ}.\,41'.\,32''{\cdot}9532.$

(9) $170^{g}.\,63'.\,27'' = 170^{g}{\cdot}6327$

$$\begin{array}{lr} & 9 \\ 10 \,| & 1535{\cdot}6943 \\ \text{degrees} & 153{\cdot}56943 \\ & 60 \\ \text{minutes} & 34{\cdot}16580 \\ & 60 \\ \text{seconds} & 9{\cdot}94800 \end{array}$$

$\therefore\ 170^{g}.\,63'.\,27'' = 153^{\circ}.\,34'.\,9''{\cdot}948.$

(10) $324^{g}.\,13'.\,88''{\cdot}7 = 324^{g}{\cdot}13887$

$$\begin{array}{lr} & 9 \\ 10 \,| & 2917{\cdot}24983 \\ \text{degrees} & 291{\cdot}724983 \\ & 60 \\ \text{minutes} & 43{\cdot}498980 \\ & 60 \\ \text{seconds} & 29{\cdot}938800 \end{array}$$

$\therefore\ 324^{g}.\,13'.\,88''{\cdot}7 = 291^{\circ}.\,43'.\,29''{\cdot}9388.$

EXAMPLES—VIII. (p. 21).

(1) Circular measure is $\frac{60\times\pi}{180}=\frac{\pi}{3}$.

(2) Circular measure is $\frac{22\cdot5\times\pi}{180}=\frac{\pi}{8}$.

(3) Circular measure is $\frac{11\cdot25\times\pi}{180}=\frac{\pi}{16}$.

(4) Circular measure is $\frac{270\times\pi}{180}=\frac{3\pi}{2}$.

(5) Circular measure is $\frac{315\times\pi}{180}=\frac{7\pi}{4}$.

(6) Circular measure is $\frac{24\frac{13}{60}\times\pi}{180}=\frac{1453\pi}{60\times180}=\frac{1453\pi}{10800}$.

(7) Circular measure is $\frac{95\frac{1}{3}\times\pi}{180}=\frac{286\times\pi}{180\times3}=\frac{143\pi}{270}$.

(8) Circular measure is $\frac{12\frac{304}{3600}\times\pi}{180}=\frac{43504\times\pi}{180\times3600}=\frac{2719\pi}{40500}$.

(9) Circular measure of each angle is $\frac{60\times\pi}{180}=\frac{\pi}{3}$.

(10) The angles are 90°, 45°, 45°, and of these the circular measures are $\frac{\pi}{2}$, $\frac{\pi}{4}$, $\frac{\pi}{4}$.

EXAMPLES—IX. (p. 22).

(1) Measure in degrees is $\frac{\pi\times180}{2\times\pi}=90$.

(2) Measure in degrees is $\frac{\pi\times180}{3\times\pi}=60$.

(3) Measure in degrees is $\frac{\pi\times180}{4\times\pi}=45$.

(4) Measure in degrees is $\frac{\pi \times 180}{6 \times \pi} = 30.$

(5) Measure in degrees is $\frac{2\pi \times 180}{3 \times \pi} = 120.$

(6) Measure in degrees is $\frac{1 \times 180}{2 \times \pi} = \frac{90}{\pi}.$

(7) Measure in degrees is $\frac{1 \times 180}{3 \times \pi} = \frac{60}{\pi}.$

(8) Measure in degrees is $\frac{1 \times 180}{4 \times \pi} = \frac{45}{\pi}.$

(9) Measure in degrees is $\frac{1 \times 180}{6 \times \pi} = \frac{30}{\pi}.$

(10) Measure in degrees is $\frac{2 \times 180}{3 \times \pi} = \frac{120}{\pi}.$

Examples—X. (p. 22).

(1) Circular measure is $\frac{50 \times \pi}{200} = \frac{\pi}{4}.$

(2) Circular measure is $\frac{25 \times \pi}{200} = \frac{\pi}{8}.$

(3) Circular measure is $\frac{6{\cdot}25 \times \pi}{200} = \frac{\pi}{32}.$

(4) Circular measure is $\frac{250 \times \pi}{200} = \frac{5\pi}{4}.$

(5) Circular measure is $\frac{500 \times \pi}{200} = \frac{5\pi}{2}.$

(6) Circular measure is $\frac{13{\cdot}0505 \times \pi}{200} = {\cdot}0652525\pi.$

(7) Circular measure is $\frac{24{\cdot}150215 \times \pi}{200} = {\cdot}120751075\pi.$

(8) Circular measure is $\frac{125{\cdot}0013 \times \pi}{200} = {\cdot}6250065\pi.$

(9) Circular measure is $\frac{\cdot 03 \times \pi}{200} = \cdot 00015\pi$.

(10) Circular measure is $\frac{\cdot 0005 \times \pi}{200} = \cdot 0000025\pi$.

EXAMPLES—XI. (p. 22).

(1) Measure in grades is $\frac{\pi \times 200}{3 \times \pi} = 66 \cdot \dot{6}$.

(2) Measure in grades is $\frac{\pi \times 200}{5 \times \pi} = 40$.

(3) Measure in grades is $\frac{\pi \times 200}{6 \times \pi} = 33 \cdot \dot{3}$.

(4) Measure in grades is $\frac{2\pi \times 200}{3 \times \pi} = 133 \cdot \dot{3}$.

(5) Measure in grades is $\frac{3\pi \times 200}{5 \times \pi} = 120$.

(6) Measure in grades is $\frac{1 \times 200}{3 \times \pi} = \frac{200}{3\pi}$.

(7) Measure in grades is $\frac{1 \times 200}{5 \times \pi} = \frac{40}{\pi}$.

(8) Measure in grades is $\frac{1 \times 200}{8 \times \pi} = \frac{25}{\pi}$.

(9) Measure in grades is $\frac{3 \times 200}{5 \times \pi} = \frac{120}{\pi}$.

(10) Measure in grades is $\frac{23 \times 200}{10 \times \pi} = \frac{460}{\pi}$.

EXAMPLES—XII. (p. 23).

(1) Measure $= 22\frac{1}{2} \div 5 = 22 \cdot 5 \div 5 = 4 \cdot 5$.

(2) Unit $= 42 \cdot 5^\circ \div 10 = 4 \cdot 25^\circ$.

(3) Angle $= 8 \times 2^\circ$, or, 16°;

$\therefore$ larger unit $= 16^\circ \div 5 = 3\frac{1}{5}^\circ$.

Then, smaller unit in terms of larger is $2 \div 3\frac{1}{5}$, or, $\frac{5}{8}$,

and larger unit in terms of smaller is $3\frac{1}{5} \div 2$, or, $\frac{8}{5}$.

(4) Angle $=7\times 3^\circ$, or, 21°;

$$\therefore \text{ larger unit} = 21^\circ \div 6 = 3\tfrac{1}{2}^\circ.$$

Then, smaller unit in terms of larger is $3 \div 3\frac{1}{2}$, or, $\frac{6}{7}$,
and larger unit in terms of smaller is $3\frac{1}{2} \div 3$, or, $\frac{7}{6}$.

(5) Measure $=42 \div 45 = \frac{14}{15}$.

(6) $13^\circ.13'.48'' = 47628''$,

$$14^g.7' = \frac{227934''}{5};$$

$\therefore$ ratio $=47628\times 5 : 227934 = 70 : 67$.

(7) $G : D = 10 : 9$, $\therefore$, $9G = 10D$, $\therefore$, $G = D + \frac{1}{9}D$.

(8) The angles of each triangle are 90°, 60°, 30°, because the line, drawn from any angle of an equilateral triangle to bisect the base, cuts the base at right angles, and bisects the vertical angle.

Expressed in grades the angles are 100^g, $66\frac{2}{3}^g$, $33\frac{1}{3}^g$.

(9) Let $x+y$, x, $x-y$ be the angles.

Then $x+y+x+x-y = 180^\circ$, or, $3x = 180^\circ$, or, $x = 60^\circ$.

(10)

$$\begin{array}{lr}
 & 39^g\cdot 012 \\
 & 9 \\ \hline
10 \,\big|\, & 351\cdot 108 \\ \hline
\text{degrees} & 35\cdot 1108 \\
 & 60 \\ \hline
\text{minutes} & 6\cdot 6480 \\
 & 60 \\ \hline
\text{seconds} & 38\cdot 8800
\end{array}$$

(11) Number of degrees in the angle $=\dfrac{m}{60}$.

Number of grades „ $=\dfrac{m\times 10}{60\times 9}$

Number of French seconds „ $=\dfrac{m\times 10\times 100\times 100}{60\times 9} = \dfrac{5000m}{27}$.

(12) $5^\circ.33'.20'' = 20000''$; and $90^\circ = 324000''$;

$$\therefore \text{ fraction} = \frac{20000}{5} \div 324000 = \frac{4000}{324000} = \frac{1}{81}.$$

(13) Let x be the measure of the angle in degrees.

Then $\frac{10x}{9}$ is the measure of the angle in grades,

and $\frac{1}{x}+\frac{9}{10x}=1$, or, $10x=19$, or, $x=1{\cdot}9$;

$\therefore$ unit angle is $1{\cdot}9°$.

(14) Let $x+y$, x, $x-y$ be the angles expressed in degrees.

Then $\frac{10(x+y)}{9}=x+(x-y)$;

or, $10x+10y=18x-9y$, and $\therefore x=\frac{19y}{8}$;

$\therefore$ the angles are $\frac{27y}{8}, \frac{19y}{8}, \frac{11y}{8}$,

and these are in the ratio 27 : 19 : 11.

(15) $\frac{180°}{\sqrt{3}}=\frac{10\times180^{g}}{9\times\sqrt{3}}=\frac{200^{g}}{\sqrt{3}}=\frac{200\sqrt{3}^{g}}{3}=115^{g}.47'$ nearly.

(16) Let $x+y$, x, $x-y$ be the angles expressed in degrees.
Then $x+y+x+x-y=180°$, or, $3x=180°$, or, $x=60°$.
Also $\frac{10}{9}(60-y):60+y=2:9$;
or, $600-10y=120+2y$, and $\therefore y=40°$.
Hence the angles are 100°, 60°, 20°.

(17) Circumference : diameter $=360:2\times57{\cdot}29577$
$=180:57{\cdot}29577$
$=3{\cdot}14159\ldots:1$.

(18) The sum of the two angles is 90°, because the third angle is 90°.
Hence, dividing 90° into two parts proportional to 2 and 3, we have 36° and 54° for the angles.
$\therefore$ angles expressed in degrees are 90°, 54°, 36°.
" " grades are 100^{g}, 60^{g}, 40^{g}.
" " circular measure are $\frac{\pi}{2}, \frac{3\pi}{10}, \frac{\pi}{5}$.

(19) Angle : $360°=13:27$;
$\therefore$ angle $=\frac{360\times13}{27}$ degrees $=\frac{40\times13}{3}$ degrees $=173\frac{1}{3}°$.

(20) Angle : $400^g = 17 : 54$;

$\therefore$ angle $= \dfrac{400 \times 17}{54}$ grades $= 125 \cdot \dot{9}2\dot{5}$ grades.

(21) Angle subtended by an arc 18 inches long = unit of circular

$$\text{measure} = \frac{200}{\pi} \text{ grades};$$

$\therefore$ angle subtended by an arc 24 inches long $= \dfrac{24 \times 200}{18 \times \pi}$ gr. $= \dfrac{800}{3\pi}$ gr.

(22) 1st angle contains $\dfrac{2 \times 200}{\pi}$ grades, or, $\dfrac{400}{\pi}$ grades.

2d angle contains $\dfrac{10 \times 20}{9}$ grades, or, $\dfrac{200}{9}$ grades.

3d angle contains $\left(200 - \dfrac{400}{\pi} - \dfrac{200}{9}\right)$gr., or, $\dfrac{1600\pi - 3600}{9\pi}$ gr.

(23) Angle required $= \dfrac{7}{2}$ of $15^\circ. 39'. 7'' = 54^\circ. 46'. 54'' \cdot 5$.

(24) Circular measure $= \dfrac{11 \cdot 3 \times \pi}{200} = \dfrac{113 \times 355}{2000 \times 113} = \dfrac{71}{400} = \cdot 1775$.

(25) Measure in degrees $= \dfrac{180 \times \pi^2}{\pi \times 9} = 20\pi$.

(26) Larger circumference = 400 times smaller circumference.
Then, since $\frac{1}{360}$th part of smaller circumference subtends an angle of 1° at the centre, it follows that $\frac{1}{400}$ of $\frac{1}{360}$th part of the larger circumference will subtend the same angle.

$$\therefore \text{ part required} = \frac{1}{400 \times 360} = \frac{1}{144000}.$$

(27) 4 right angles $= 360^\circ = 400^g = 2\pi^c$;

$\therefore$ the measure of 1° will be $\dfrac{1}{360}$,

the measure of 1^g will be $\dfrac{1}{400}$,

the measure of 1^c will be $\dfrac{1}{2\pi}$.

(28) Length of whole circumference of earth $= 7980\pi$ miles ;

$\therefore$ length of 1 degree of meridian $= \frac{7980\pi}{360}$ miles $= \frac{133\pi}{6}$ miles.

(29) (1) $\frac{3}{2} \times 45° = 67\frac{1}{2}°$; $4 \times 45° = 180°$; $\pi \times 45° = 45\pi°$;

$$\left(4n + \frac{1}{3}\right) \times 45° = (n \,.\, 180 + 15)°.$$

(2) $\frac{3}{2} \times \frac{\pi}{4} = \frac{3\pi}{8}$; $4 \times \frac{\pi}{4} = \pi$; $\pi \times \frac{\pi}{4} = \left(\frac{\pi}{2}\right)^2$;

$$\left(4n + \frac{1}{3}\right) \times \frac{\pi}{4} = n\pi + \frac{\pi}{12}.$$

(30) Number of degrees in the unit angle $= \frac{3 \times 180}{\pi}$;

$\therefore$ measure of an angle of $45° = 45 \div \frac{3 \times 180}{\pi} = \frac{45 \times \pi}{3 \times 180} = \frac{\pi}{12}$.

(31) (1) Sum of angles $= (12 - 4)$ right angles $= 8$ right angles.
(EUCLID, I. XXXII., COR. 1.)

$\therefore$ each angle $= \frac{8 \times 90}{6}$ degrees $= 120°$.

(2) Sum of angles $= (10 - 4)$ right angles $= 6$ right angles ;

$\therefore$ each angle $= \frac{6 \times 90}{5}$ degrees $= 108°$.

(32) (1) Each angle $= \frac{6 \times 100}{5}$ grades $= 120^g$.

(2) Each angle $= \frac{12 \times 100}{8}$ grades $= 150^g$.

(33) (1) Circular measure of each angle $= \frac{\pi}{3}$.

(2) Circular measure of each angle $= \frac{8 \times \pi}{6 \times 2} = \frac{2\pi}{3}$.

(34) Sum of all the angles $= (2n - 4)$ right angles ;

$\therefore$ circular measure of each angle $= \frac{2n - 4}{n} \cdot \frac{\pi}{2} = \pi - \frac{2\pi}{n}$.

(35) Arc subtending an angle of $180° = 18\pi$ feet.

$\therefore$ arc subtending an angle of $10° = \frac{18\pi}{18}$ feet $= \pi$ feet.

(36) Let $2n$ and n be the number of sides in the polygons, respectively.

Each angle in first polygon contains $\frac{4n-4}{2n}$ right angles.

Each angle in second polygon contains $\frac{2n-4}{n}$ right angles.

$$\therefore \frac{4n-4}{2n} : \frac{2n-4}{n} = 3 : 2;$$

$$\therefore 4n-4=6n-12, \text{ or, } 2n=8, \text{ or, } n=4.$$

Hence the number of sides will be 8 and 4 respectively.

Examples—XIII. (p. 36).

(1) $\sin BAD=\frac{BD}{AB}$; $\cos BAD=\frac{AD}{AB}$; $\tan BAD=\frac{BD}{AD}$;

$\sin ABD=\frac{AD}{AB}$; $\cot ABD=\frac{BD}{AD}$; $\operatorname{cosec} ABD=\frac{AB}{AD}$;

$\sin BCD=\frac{BD}{BC}$; $\sin CBD=\frac{CD}{BC}$; $\tan BCD=\frac{DB}{DC}$.

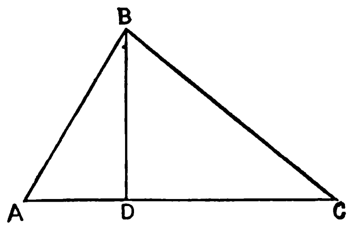

Fig. 6.

(2) $\frac{a}{b}=\sin A$, $\therefore a=b.\sin A$,

$\frac{a}{b}=\cos C$, $\therefore a=b.\cos C$,

$\frac{a}{c}=\tan A$, $\therefore a=c.\tan A$,

$\frac{a}{c}=\cot C$, $\therefore a=c.\cot C$;

and similarly for the rest of the Examples.

EXAMPLES—XIV. (p. 49).

(1) $\cos\alpha . \sin\gamma . \cos\delta = \cos0° . \sin45° . \cos60° = 1 \times \frac{1}{\sqrt{2}} . \frac{1}{2} = \frac{1}{2\sqrt{2}}$.

(2) $\sin\theta . \cos\frac{\pi}{4} . \operatorname{cosec}\delta = \sin90° . \cos45° . \operatorname{cosec}60°$

$$= 1 \times \frac{1}{\sqrt{2}} \times \frac{2}{\sqrt{3}} = \sqrt{\frac{2}{3}}.$$

(3) $\sin\frac{\pi}{2} + \cos\frac{\pi}{6} - \sec\alpha = \sin90° + \cos30° - \sec0° = 1 + \frac{\sqrt{3}}{2} - 1 = \frac{\sqrt{3}}{2}$.

(4) $\sin\frac{\pi}{3} . \operatorname{cosec}\frac{\pi}{2} . \sec\delta = \sin60° . \operatorname{cosec}90° . \sec60° = \frac{\sqrt{3}}{2} \times 1 \times 2 = \sqrt{3}$.

(5) $(\sin\theta - \cos\theta + \operatorname{cosec}\beta)\left(\cos\theta + \sec\frac{\pi}{4} + \cot\delta\right)$

$$= (\sin90° - \cos90° + \operatorname{cosec}30°).(\cos90° + \sec45° + \cot60°)$$

$$= (1 - 0 + 2).\left(0 + \sqrt{2} + \frac{1}{\sqrt{3}}\right) = 3 \times \left(\sqrt{2} + \frac{1}{\sqrt{3}}\right) = 3\sqrt{2} + \sqrt{3}.$$

(6) $(\sin\delta - \sin\gamma)(\cos\beta + \cos\gamma) = (\sin60° - \sin45°)(\cos30° + \cos45°)$

$$= \left(\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\right) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}.$$

$$\sin^2\beta = \sin^2 30° = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

(7) $\cot^2\frac{\pi}{4} - \cot^2\frac{\pi}{6} = \cot^2 45° - \cot^2 30° = 1 - 3 = -2$.

$$\frac{\sin^2\frac{\pi}{6} - \sin^2\frac{\pi}{4}}{\sin^2\frac{\pi}{4} . \sin^2\frac{\pi}{6}} = \frac{\frac{1}{4} - \frac{1}{2}}{\frac{1}{4} . \frac{1}{2}} = \frac{-2}{1} = -2.$$

(8) $\left(\sin\frac{\pi}{6}+\cos\frac{\pi}{6}\right)\left(\sin\frac{\pi}{3}-\cos\frac{\pi}{3}\right)=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)$

$$=\frac{3}{4}-\frac{1}{4}=\frac{1}{2}=\cos\frac{\pi}{3}.$$

(9) $\cos\frac{\pi}{3}\cdot\cos\frac{\pi}{6}=\frac{1}{2}\cdot\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{4}.$

$$\frac{1}{2}\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)+\frac{1}{2}\cos\left(\frac{\pi}{3}-\frac{\pi}{6}\right)=\frac{1}{2}\cos\frac{\pi}{2}+\frac{1}{2}\cos\frac{\pi}{6}$$

$$=\frac{1}{2}\times 0+\frac{1}{2}\times\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{4}.$$

(10) $\tan^2\frac{\pi}{3}-\tan^2\frac{\pi}{6}=3-\frac{1}{3}=\frac{8}{3}.$

$$\frac{\sin^2\frac{\pi}{3}-\sin^2\frac{\pi}{6}}{\cos^2\frac{\pi}{3}\cdot\cos^2\frac{\pi}{6}}=\frac{\frac{3}{4}-\frac{1}{4}}{\frac{1}{4}\cdot\frac{3}{4}}=\frac{8}{3}.$$

EXAMPLES—XV. (p. 52).

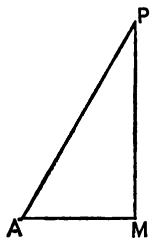

FIG. 7.

(1.) Let PM be the tower; A the place of observation.

Then $AM=200$ feet, and $\angle PAM=60°$.

Now $PM=AM.\tan PAM$

$=(200\times\sqrt{3})$ feet $=346{\cdot}4101\ldots$ feet.

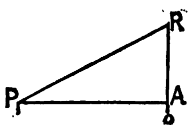

FIG. 8.

(2) Let RO be the tower; P the point of observation.

Then $AP=140$ feet, and $\angle RPA=30°$.

Now $RA=PA.\tan 30°$.

$=\frac{140}{\sqrt{3}}$ feet $=\frac{140\sqrt{3}}{3}$ feet $=80{\cdot}829037\ldots$ feet.

$\therefore$ $RO=80{\cdot}829037\ldots$ feet $+5$ feet $=85{\cdot}829037\ldots$ feet.

(3) Taking the diagram in Art. 87,

$$AB : BQ = \sqrt{3} : 1;$$

$$\therefore \tan SQR = \sqrt{3}, \text{ and } \therefore \angle SQR = 60^\circ.$$

(4) Let AB be the steeple; P the point of observation.

Then $PB = 300$ feet, and $\angle APB = 30^\circ$.

Then $AB = PB \cdot \tan APB$

$$= 300 \cdot \frac{1}{\sqrt{3}} \text{ feet} = 100\sqrt{3} \cdot \text{feet}$$

$$= 173{\cdot}205 \ldots \text{ feet.}$$

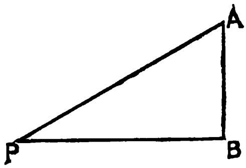

FIG. 9.

(5) Let AP be the rock; O the position of the ship.

Then $AP = 245$ feet; and $\angle AOP = 30^\circ$.

Now $PO = AP \cdot \cot AOP$

$$= 245 \cdot \sqrt{3} \text{ ft.} = 424{\cdot}352 \ldots \text{ ft.}$$

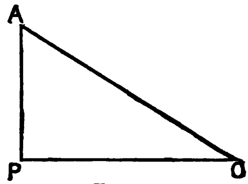

FIG. 10.

(6) Let AB be the hill; C and D the positions of the milestones.

Then $DC = 1$ mile; $\angle ACB = 45^\circ$;

$\angle ADB = 30^\circ$.

Hence $\angle CAB = 45^\circ$, and $AB = BC$.

Let x = height of hill in miles.

Then $AB = BD \cdot \tan ADB$

$$= (BC + CD) \cdot \tan 30^\circ;$$

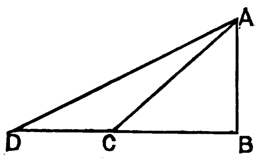

FIG. 11.

$$\therefore x = (x+1) \cdot \frac{1}{\sqrt{3}};$$

$$\therefore \sqrt{3} \cdot x = x + 1, \text{ or, } x = \frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{3-1} = \frac{\sqrt{3}+1}{2};$$

$$\therefore x = \frac{2{\cdot}732 \ldots}{2} \text{ miles} = 1{\cdot}366 \ldots \text{ miles.}$$

(7) Let AO be the flag-staff; PO the tower; M the point of observation.

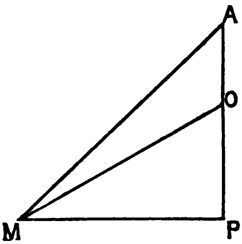

FIG. 12.

Then $PM=100$ feet; $\angle AMP=45°$; $\angle OMP=30°$.

Then $AO=AP-OP$

$$=PM-PM.\tan OMP$$

$$=\left(100-100\cdot\frac{1}{\sqrt{3}}\right)\text{ feet.}$$

$$=\frac{300-100\sqrt{3}}{3}\text{ feet}$$

$$=\frac{300-173\cdot205\ldots}{3}\text{ feet}=42\cdot265\ldots\text{ feet.}$$

(8) Let AP be the tower; MO the column; MD parallel to OP.

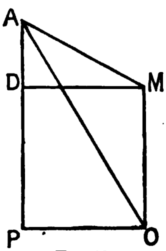

FIG. 13.

Then $\angle AMD=30°$, and $\angle AOP=60°$.

Then $MD=OP=AP.\cot AOP=\frac{108}{\sqrt{3}}$ feet.

And $AD=MD.\tan AMD=\frac{108}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}$ feet

$$=\frac{108}{3}\text{ feet}=36\text{ feet.}$$

Hence $OM=AP-AD=(108-36)$ feet$=72$ feet.

(9.) Let AP be the tower; M and O the points of observation.

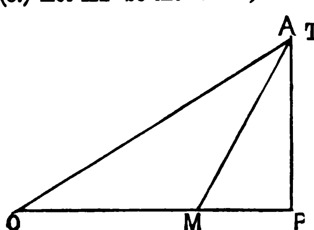

FIG. 14.

Then $\angle AMP=60°$; and $\angle AOP=30°$.

Let $x=$ height of tower in yards.

Now $\angle AMP=\angle AOP+\angle OAM$,

or, $60°=30°+\angle OAM$;

$\therefore\ \angle OAM=30°=\angle AOM$,

and $\therefore\ MA=OM=100$ yards.

Then $AP=MA.\sin AMP=100\cdot\frac{\sqrt{3}}{2}$ yards$=50\sqrt{3}$ yards.

(10) Taking the diagram in Art. 87.

$$\tan AQB = \frac{AB}{BQ} = \frac{10}{25} = \cdot 4\,;$$

$\therefore$ altitude of the sun is 25°.

(11) The diagram represents a vertical section of the spire and tower.

Let x represent the height of the spire in feet.

Then $AM = x + 35 - 23 = x + 12$,

$BM = 60 + 17\frac{1}{2} = 77 \cdot 5$,

and $\frac{x+12}{77 \cdot 5} = \tan ABM = 1 \cdot 5\,;$

$\therefore\ x + 12 = 116 \cdot 25$, or, $x = 104 \cdot 25$ feet.

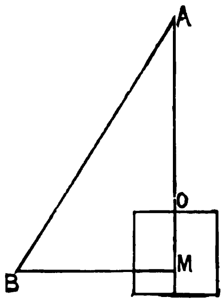

FIG. 15.

(12) Let OB be the height of the kite in yards.

Let AO be the string.

Then $OB = AO \,.\, \sin OAB$

$= \left(250 \times \frac{1}{2}\right)$ yards $= 125$ yards.

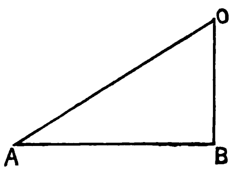

FIG. 16.

(13) Let AC be the rope; AB the height of the house.

Then $\angle\, ACB = 40^\circ \,.\, 30'$.

And $AC = \frac{AB}{\sin ACB} = \frac{60}{\cdot 65}$ feet $= 92\frac{4}{13}$ feet.

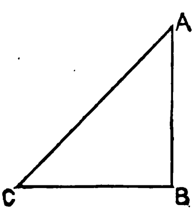

FIG. 17.

(14) Let AC be the tower; BC the breadth of the river.

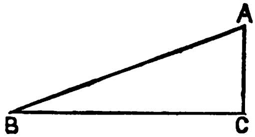

Fig. 18.

Then $\angle ABC = 20^\circ$.

And $BC = \dfrac{AC}{\tan ABC}$

$= \dfrac{120}{\cdot 35}$ feet $= 342\frac{6}{7}$ feet.

(15) Taking the diagram of Art. 87.

Length of shadow $= QB = \dfrac{AB}{\tan AQB} = \dfrac{6}{\cdot 745}$ feet $= 8\cdot 053 \ldots$ feet.

Examples—XVI. (p. 57).

(1) $\cos\theta \cdot \tan\theta = \cos\theta \cdot \dfrac{\sin\theta}{\cos\theta} = \sin\theta.$

(2) $\sin\theta \cdot \cot\theta = \sin\theta \cdot \dfrac{\cos\theta}{\sin\theta} = \cos\theta.$

(3) $\sin a \cdot \sec a = \sin a \cdot \dfrac{1}{\cos a} = \dfrac{\sin a}{\cos a} = \tan a.$

(4) $\cos a \cdot \operatorname{cosec} a = \cos a \cdot \dfrac{1}{\sin a} = \dfrac{\cos a}{\sin a} = \cot a.$

(5) $(1 + \tan^2\theta) \cdot \cos^2\theta = \sec^2\theta \cdot \cos^2\theta = \dfrac{\cos^2\theta}{\cos^2\theta} = 1.$

(6) $(1 + \cot^2\theta) \cdot \sin^2\theta = \operatorname{cosec}^2\theta \cdot \sin^2\theta = \dfrac{\sin^2\theta}{\sin^2\theta} = 1.$

(7) $\dfrac{\tan^2 a}{1 + \tan^2 a} = \dfrac{\tan^2 a}{\sec^2 a} = \dfrac{\sin^2 a}{\cos^2 a} \cdot \cos^2 a = \sin^2 a.$

(8) $\dfrac{\operatorname{cosec}^2 a - 1}{\operatorname{cosec}^2 a} = 1 - \dfrac{1}{\operatorname{cosec}^2 a} = 1 - \sin^2 a = \cos^2 a.$

(9) $\tan x + \cot x = \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x} = \dfrac{\sin^2 x + \cos^2 x}{\cos x \cdot \sin x} = \dfrac{1}{\cos x \cdot \sin x} = \sec x \cdot \operatorname{cosec} x.$

(10) $\dfrac{\cos x \,.\, \operatorname{cosec} x \,.\, \tan x}{\sin x \,.\, \sec x \,.\, \cot x} = \dfrac{\cos x \cdot \dfrac{1}{\sin x} \cdot \dfrac{\sin x}{\cos x}}{\sin x \cdot \dfrac{1}{\cos x} \cdot \dfrac{\cos x}{\sin x}} = \dfrac{\cos x \,.\, \sin x \,.\, \cos x \,.\, \sin x}{\sin x \,.\, \sin x \,.\, \cos x \,.\, \cos x} = 1.$

(11) $\cos x + \sin x \,.\, \tan x = \cos x + \dfrac{\sin^2 x}{\cos x} = \dfrac{\cos^2 x + \sin^2 x}{\cos x} = \dfrac{1}{\cos x} = \sec x.$

(12) $\dfrac{\cos\theta}{\tan\theta \,.\, \cot^2\theta} = \dfrac{\cos\theta}{\cot\theta} = \dfrac{\cos\theta \,.\, \sin\theta}{\cos\theta} = \sin\theta.$

(13) $(\cos^2\theta - 1)(\cot^2\theta + 1) = (\cos^2\theta - 1) \,.\, \operatorname{cosec}^2\theta = -\sin^2\theta \times \dfrac{1}{\sin^2\theta} = -1.$

(14) $\cot^2 a - \cos^2 a = \dfrac{\cos^2 a}{\sin^2 a} - \cos^2 a = \cos^2 a\left(\dfrac{1}{\sin^2 a} - 1\right) = \cos^2 a \cdot \dfrac{1 - \sin^2 a}{\sin^2 a}$

$= \cos^2 a \cdot \dfrac{\cos^2 a}{\sin^2 a} = \cot^2 a \,.\, \cos^2 a.$

(15) $\sec^2 a \,.\, \operatorname{cosec}^2 a = \sec^2 a(1 + \cot^2 a) = \sec^2 a + \sec^2 a \cdot \dfrac{\cos^2 a}{\sin^2 a}$

$= \sec^2 a + \operatorname{cosec}^2 a.$

(16) $\sin^2\phi + \sin^2\phi \,.\, \tan^2\phi = \sin^2\phi(1 + \tan^2\phi) = \sin^2\phi \,.\, \sec^2\phi = \tan^2\phi.$

(17) $\cot^2\phi \,.\, \sin^2\phi + \sin^2\phi = \sin^2\phi(\cot^2\phi + 1) = \sin^2\phi \,.\, \operatorname{cosec}^2\phi = 1.$

(18) $\sec^2\phi - 1 = \tan^2\phi = \dfrac{\sin^2\phi}{\cos^2\phi} = \sin^2\phi \,.\, \sec^2\phi.$

(19) $2\operatorname{versin}\phi - \operatorname{versin}^2\phi = 2(1 - \cos\phi) - (1 - \cos\phi)^2$

$= 2 - 2\cos\phi - 1 + 2\cos\phi - \cos^2\phi = 1 - \cos^2\phi = \sin^2\phi.$

(20) $\dfrac{\sec\theta - 1}{\sec\theta} = 1 - \dfrac{1}{\sec\theta} = 1 - \cos\theta = \operatorname{versin}\theta.$

EXAMPLES—XVII. (p. 60).

(1) Let PAM be an angle whose cosine is c.
Draw PM perpendicular to AM.
Then if AP be represented by 1,
AM will be represented by c, and
PM will be represented by $\sqrt{1-c^2}$.
Then, denoting $\angle PAM$ by A,

$\sin A = \dfrac{PM}{AP} = \dfrac{\sqrt{1-c^2}}{1} = \sqrt{1-\cos^2 A}$

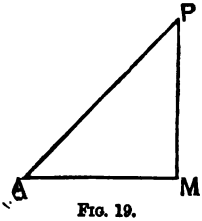

FIG. 19.

$$\tan A = \frac{PM}{AM} = \frac{\sqrt{1-c^2}}{c} = \frac{\sqrt{1-\cos^2 A}}{\cos A}$$

$$\sec A = \frac{AP}{AM} = \frac{1}{c} = \frac{1}{\cos A}$$

$$\operatorname{cosec} A = \frac{AP}{PM} = \frac{1}{\sqrt{1-c^2}} = \frac{1}{\sqrt{1-\cos^2 A}}$$

$$\cot A = \frac{AM}{PM} = \frac{c}{\sqrt{1-c^2}} = \frac{\cos A}{\sqrt{1-\cos^2 A}}$$

(2) Let PAM be an angle whose cosecant is c.

Constructing a diagram as in Example (1), the measures of AP, PM, AM may be taken as c, 1, $\sqrt{c^2-1}$ respectively.

$$\text{Then } \sin A = \frac{PM}{AP} = \frac{1}{c} = \frac{1}{\operatorname{cosec} A}$$

$$\cos A = \frac{AM}{AP} = \frac{\sqrt{c^2-1}}{c} = \frac{\sqrt{\operatorname{cosec}^2 A - 1}}{\operatorname{cosec} A}$$

$$\tan A = \frac{PM}{AM} = \frac{1}{\sqrt{c^2-1}} = \frac{1}{\sqrt{\operatorname{cosec}^2 A - 1}}$$

$$\sec A = \frac{AP}{AM} = \frac{c}{\sqrt{c^2-1}} = \frac{\operatorname{cosec} A}{\sqrt{\operatorname{cosec}^2 A = 1}}$$

$$\cot A = \frac{AM}{MP} = \frac{\sqrt{c^2-1}}{1} = \sqrt{\operatorname{cosec}^2 A - 1}.$$

(3) Let PAM be an angle whose secant is s.

Constructing a diagram as in Example (1), the measures of AP, AM, PM, may be taken as s, 1, $\sqrt{s^2-1}$ respectively.

$$\text{Then } \sin A = \frac{PM}{AP} = \frac{\sqrt{s^2-1}}{s} = \frac{\sqrt{\sec^2 A - 1}}{\sec A}$$

$$\cos A = \frac{AM}{AP} = \frac{1}{s} = \frac{1}{\sec A}$$

$$\tan A = \frac{PM}{AM} = \frac{\sqrt{s^2-1}}{1} = \sqrt{\sec^2 A - 1}$$

$$\operatorname{cosec} A = \frac{AP}{PM} = \frac{s}{\sqrt{s^2-1}} = \frac{\sec A}{\sqrt{\sec^2 A - 1}}$$

$$\cot A = \frac{AM}{PM} = \frac{1}{\sqrt{s^2-1}} = \frac{1}{\sqrt{\sec^2 A - 1}}.$$

(4) Let PAM be an angle whose cotangent is c.

Constructing a diagram as in Example (1), the measures of AM, PM, AP may be taken as c, 1, $\sqrt{1+c^2}$ respectively.

$$\text{Then } \sin A = \frac{PM}{AP} = \frac{1}{\sqrt{1+c^2}} = \frac{1}{\sqrt{1+\cot^2 A}}$$

$$\cos A = \frac{AM}{AP} = \frac{c}{\sqrt{1+c^2}} = \frac{\cot A}{\sqrt{1+\cot^2 A}}$$

$$\tan A = \frac{PM}{AM} = \frac{1}{c} = \frac{1}{\cot A}$$

$$\operatorname{cosec} A = \frac{AP}{PM} = \frac{\sqrt{1+c^2}}{1} = \sqrt{1+\cot^2 A}$$

$$\sec A = \frac{AP}{AM} = \frac{\sqrt{1+c^2}}{c} = \frac{\sqrt{1+\cot^2 A}}{\cot A}.$$

Examples—XVIII. (p. 61).

(1) Take the diagram as before; then if $\angle PAM$ be denoted by a, the measure of PM may be denoted by 2, the measure of AP by 3, and therefore the measure of AM by $\sqrt{9-4} = \sqrt{5}$.

$$\text{Then } \cos a = \frac{\sqrt{5}}{3} \text{ and } \tan a = \frac{2}{\sqrt{5}}.$$

(2) Let the measure of AM be 4, and that of AP be 5; then that of AM will be $\sqrt{25-16}$, or, 3.

$$\text{Then } \sin a = \frac{3}{5}, \text{ and } \tan a = \frac{3}{4}.$$

(3) Let the measure of AP be 4, and that of PM be 3; then that of PM will be $\sqrt{16-9}$, or, $\sqrt{7}$.

$$\text{Then } \cos\theta = \frac{\sqrt{7}}{4}, \text{ and } \tan\theta = \frac{3}{\sqrt{7}}.$$

(4) Let the measure of PM be 1, and that of AP be $\sqrt{3}$; then that of AM will be $\sqrt{3-1}$, or, $\sqrt{2}$.

$$\text{Then } \cos\theta = \sqrt{\frac{2}{3}}, \text{ and } \tan\theta = \frac{1}{\sqrt{2}}.$$

(5) Let the measure of PM be a^2, and that of AM be b^2; then that of AP will be $\sqrt{a^4+b^4}$.

$$\text{Then } \operatorname{cosec} a=\frac{\sqrt{a^4+b^4}}{a^2}, \text{ and } \sec a=\frac{\sqrt{a^4+b^4}}{b^2}.$$

(6) Let the measure of AM be a, and that of AP be b; then that of PM will be $\sqrt{b^2-a^2}$.

$$\text{Then } \tan a=\frac{\sqrt{b^2-a^2}}{a}, \text{ and } \operatorname{cosec} a=\frac{b}{\sqrt{b^2-a^2}}.$$

(7) Let the measure of PM be a, and that of AP be 1; then that of AM will be $\sqrt{1-a^2}$.

$$\text{Then } \tan\theta=\frac{a}{\sqrt{1-a^2}}, \text{ and } \sec\theta=\frac{1}{\sqrt{1-a^2}}.$$

(8) Let the measure of AM be b, and that of AP be 1; then that of PM will be $\sqrt{1-b^2}$.

$$\text{Then } \tan\theta=\frac{\sqrt{1-b^2}}{b}, \text{ and } \operatorname{cosec}\theta=\frac{1}{\sqrt{1-b^2}}.$$

(9) Let the measure of PM be 6, and that of AP be 10; then that of AM will be $\sqrt{100-36}$, or, 8.

$$\text{Then } \cos\theta=\frac{8}{10}=\frac{4}{5}, \text{ and } \cot\theta=\frac{8}{6}=\frac{4}{3}.$$

(10) Let the measure of AM be 5, and that of AP be 9; then that of PM will be $\sqrt{81-25}=\sqrt{56}=2\sqrt{14}$.

$$\text{Then } \cot\theta=\frac{5}{2\sqrt{14}}, \text{ and } \operatorname{cosec}\theta=\frac{9}{2\sqrt{14}}.$$

(11) Let the measure of AP be 22, and that of PM be 9; then that of AM will be $\sqrt{484-81}=\sqrt{403}$.

$$\text{Then } \cos\theta=\frac{\sqrt{403}}{22}, \text{ and } \cot\theta=\frac{\sqrt{403}}{9}.$$

(12) $1 \cdot 0\dot{3} = \frac{103-10}{90} = \frac{93}{90} = \frac{31}{30}$.

Let the measure of AP be 31, and that of AM be 30; then that of PM will be $\sqrt{961-900}$, or, $\sqrt{61}$.

$$\text{Then } \sin\theta = \frac{\sqrt{61}}{31}, \text{ and } \tan\theta = \frac{\sqrt{61}}{30}.$$

(13) Let the measure of PM be 99, and that of AP be 101; then that of AM will be $\sqrt{10201-9801}$, or, 20.

$$\text{Then } \cos\phi = \frac{20}{101}, \text{ and } \cot\phi = \frac{20}{99}.$$

(14) Let the measure of AM be 20, and that of AP be 101; then that of PM will be $\sqrt{10201-400}$, or, 99.

$$\text{Then } \sin\phi = \frac{99}{101}, \text{ and } \tan\phi = \frac{99}{20}.$$

(15) $\text{Cos}\theta = 1 - \text{versin}\theta = 1 - \frac{1}{13} = \frac{12}{13}$.

Let the measure of AM be 12, and that of AP be 13; then that of PM will be $\sqrt{169-144}$, or, 5.

$$\text{Then } \sin\theta = \frac{5}{13}, \text{ and } \sec\theta = \frac{13}{12}.$$

EXAMPLES—XIX. (p. 63).

(1) $\sin A = \frac{1}{\text{cosec} A} = \frac{1}{\sqrt{\text{cosec}^2 A}} = \frac{1}{\sqrt{(1+\cot^2 A)}}$.

(2) $\cos A = \frac{1}{\sec A} = \frac{1}{\sqrt{\sec^2 A}} = \frac{1}{\sqrt{(1+\tan^2 A)}}$.

(3) $\cos x = \frac{\cot x}{\text{cosec} x} = \frac{\cot x}{\sqrt{(\text{cosec}^2 x)}} = \frac{\cot x}{\sqrt{(1+\cot^2 x)}}$.

(4) $\tan x \,.\, \cos x = \sin x = \sqrt{(1-\cos^2 x)}$.

(5) $\cos\phi = \frac{\cot\phi}{\text{cosec}\phi} = \frac{\sqrt{(\text{cosec}^2\phi - 1)}}{\text{cosec}\phi}$.

(6) $\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{\sqrt{(1-\cos^2\phi)}}{\cos\phi} = \sqrt{\left(\frac{1-\cos^2\phi}{\cos^2\phi}\right)}$.

(7) $\sin^2\alpha = 1 - \cos^2\alpha = (1+\cos\alpha)(1-\cos\alpha) = (1+\cos\alpha) \, . \, \text{versin}\alpha$.

(8) $\tan^2\alpha - \tan^2\beta = \frac{\sin^2\alpha}{\cos^2\alpha} - \frac{\sin^2\beta}{\cos^2\beta} = \frac{\sin^2\alpha \, . \, \cos^2\beta - \cos^2\alpha \, . \, \sin^2\beta}{\cos^2\alpha \, . \, \cos^2\beta}$

$$= \frac{(1-\cos^2\alpha)\cos^2\beta - (1-\cos^2\beta)\cos^2\alpha}{\cos^2\alpha \, . \, \cos^2\beta} = \frac{\cos^2\beta - \cos^2\alpha}{\cos^2\alpha \, . \, \cos^2\beta}.$$

(9) $\cot^2\alpha - \cot^2\beta = \frac{\cos^2\alpha}{\sin^2\alpha} - \frac{\cos^2\beta}{\sin^2\beta} = \frac{\cos^2\alpha \, . \, \sin^2\beta - \cos^2\beta \, . \, \sin^2\alpha}{\sin^2\alpha \, . \, \sin^2\beta}$

$$= \frac{(1-\sin^2\alpha)\sin^2\beta - (1-\sin^2\beta)\sin^2\alpha}{\sin^2\alpha \, . \, \sin^2\beta} = \frac{\sin^2\beta - \sin^2\alpha}{\sin^2\alpha \, . \, \sin^2\beta}.$$

(10) $\sin^2\theta \, . \, \tan^2\theta + \cos^2\theta \, . \, \cot^2\theta = (1-\cos^2\theta) \, . \, \tan^2\theta + (1-\sin^2\theta) \, . \, \cot^2\theta$

$= \tan^2\theta - \sin^2\theta + \cot^2\theta - \cos^2\theta = \tan^2\theta + \cot^2\theta - (\sin^2\theta + \cos^2\theta)$

$= \tan^2\theta + \cot^2\theta - 1$.

(11) $\sec^4\theta + \tan^4\theta = (1+\tan^2\theta)^2 + \tan^4\theta = 1 + 2\tan^2\theta + \tan^4\theta + \tan^4\theta$

$$= 1 + 2\tan^2\theta(1+\tan^2\theta) = 1 + 2\tan^2\theta \, . \, \sec^2\theta.$$

(12) $\text{cosec}\theta(\sec\theta - 1) - \cot\theta(1-\cos\theta) = \frac{1}{\sin\theta . \cos\theta} - \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta} + \frac{\cos^2\theta}{\sin\theta}$

$$= \frac{1-\cos^2\theta}{\sin\theta \, . \, \cos\theta} - \frac{1-\cos^2\theta}{\sin\theta} = \frac{\sin^2\theta}{\sin\theta \, . \, \cos\theta} - \frac{\sin^2\theta}{\sin\theta} = \tan\theta - \sin\theta.$$

(13) $\cot^2 b + \tan^2 b = (\text{cosec}^2 b - 1) + (\sec^2 b - 1) = \text{cosec}^2 b + \sec^2 b - 2$

$$= \frac{1}{\sin^2 b} + \frac{1}{\cos^2 b} - 2 = \frac{\cos^2 b + \sin^2 b}{\sin^2 b \, . \, \cos^2 b} - 2.$$

$$= \frac{1}{\sin^2 b \, . \, \cos^2 b} - 2 = \text{cosec}^2 b \, . \, \sec^2 b - 2.$$

(14) $\cot^2 A - \cos^2 A = \frac{\cos^2 A}{\sin^2 A} - \cos^2 A = \cos^2 A\left(\frac{1}{\sin^2 A} - 1\right)$

$$= \cos^2 A\left(\frac{1-\sin^2 A}{\sin^2 A}\right) = \cos^2 A \, . \, \frac{\cos^2 A}{\sin^2 A} = \cos^4 A \, . \, \text{cosec}^2 A.$$

(15) $\tan^2\theta - \sin^2\theta = \frac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta = \sin^2\theta\left(\frac{1}{\cos^2\theta} - 1\right)$

$$= \sin^2\theta \, . \, \frac{1-\cos^2\theta}{\cos^2\theta} = \sin^2\theta \, . \, \frac{\sin^2\theta}{\cos^2\theta} = \sin^4\theta \, . \, \sec^2\theta.$$

(16) $(\sec\theta - \operatorname{cosec}\theta)(1 + \cot\theta + \tan\theta) = \left(\frac{1}{\cos\theta} - \frac{1}{\sin\theta}\right)\left(1 + \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}\right)$

$$= \frac{\sin\theta - \cos\theta}{\sin\theta \,.\, \cos\theta} \cdot \frac{\sin\theta \,.\, \cos\theta + 1}{\sin\theta \,.\, \cos\theta} = \frac{\sin^2\theta \,.\, \cos\theta + \sin\theta - \sin\theta \,.\, \cos^2\theta - \cos\theta}{\sin^2\theta \,.\, \cos^2\theta}$$

$$= \frac{(1 - \cos^2\theta)\cos\theta + \sin\theta - \sin\theta(1 - \sin^2\theta) - \cos\theta}{\sin^2\theta \,.\, \cos^2\theta}$$

$$= \frac{\cos\theta - \cos^3\theta + \sin\theta - \sin\theta + \sin^3\theta - \cos\theta}{\sin^2\theta \,.\, \cos^2\theta}$$

$$= \frac{\sin^3\theta - \cos^3\theta}{\sin^2\theta \,.\, \cos^2\theta} = \frac{\sin\theta}{\cos^2\theta} - \frac{\cos\theta}{\sin^2\theta} = \frac{\sec^2\theta}{\operatorname{cosec}\theta} - \frac{\operatorname{cosec}^2\theta}{\sec\theta}.$$

(17) $\frac{\operatorname{cosec}\theta}{\sec\theta} + \frac{\sec\theta}{\operatorname{cosec}\theta} = \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} = \frac{\cos^2\theta + \sin^2\theta}{\sin\theta \,.\, \cos\theta}$

$$= \frac{1}{\sin\theta \,.\, \cos\theta} = \sec\theta \,.\, \operatorname{cosec}\theta.$$

(18) $\cos\theta(\tan\theta + 2)(2\tan\theta + 1) = \cos\theta(2\tan^2\theta + 5\tan\theta + 2)$

$$= 2\cos\theta(\tan^2\theta + 1) + 5\cos\theta \,.\, \tan\theta$$

$$= 2\cos\theta \,.\, \sec^2\theta + 5 \,.\, \sin\theta = 2\sec\theta + 5\sin\theta.$$

(19) $\cos x(2\sec x + \tan x)(\sec x - 2\tan x)$

$$= \cos x(2\sec^2 x - 3\sec x \,.\, \tan x - 2\tan^2 x)$$

$$= 2\cos x(\sec^2 x - \tan^2 x) - 3\cos x \,.\, \sec x \,.\, \tan x$$

$$= 2\cos x - 3\tan x.$$

(20) $(\operatorname{cosec}\theta - \cot\theta)^2 = \operatorname{cosec}^2\theta - 2\operatorname{cosec}\theta \,.\, \cot\theta + \cot^2\theta$

$$= \frac{1}{\sin^2\theta} - \frac{2\cos\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta}$$

$$= \frac{1 - 2\cos\theta + \cos^2\theta}{\sin^2\theta} = \frac{(1 - \cos\theta)^2}{1 - \cos^2\theta}$$

$$= \frac{(1 - \cos\theta)(1 - \cos\theta)}{(1 + \cos\theta)(1 - \cos\theta)}$$

$$= \frac{1 - \cos\theta}{1 + \cos\theta}.$$

(21) $\frac{\sec\theta \,.\, \cot\theta - \operatorname{cosec}\theta \,.\, \tan\theta}{\cos\theta - \sin\theta} = \frac{\frac{1}{\sin\theta} - \frac{1}{\cos\theta}}{\cos\theta - \sin\theta} = \frac{\frac{\cos\theta - \sin\theta}{\sin\theta \,.\, \cos\theta}}{\cos\theta - \sin\theta}$

$$= \frac{1}{\sin\theta \,.\, \cos\theta} = \operatorname{cosec}\theta \,.\, \sec\theta.$$

(22) $\sec\theta + \operatorname{cosec}\theta \,.\, \tan^3\theta(1+\operatorname{cosec}^2\theta) = \dfrac{1}{\cos\theta} + \dfrac{\sin^2\theta}{\cos^3\theta} + \dfrac{1}{\cos^3\theta}$

$$= \frac{\cos^2\theta + \sin^2\theta + 1}{\cos^3\theta} = \frac{2}{\cos^3\theta} = 2\sec^3\theta.$$

(23) $(\sin\theta + \sec\theta)^2 + (\cos\theta + \operatorname{cosec}\theta)^2$

$$= \sin^2\theta + \frac{2\sin\theta}{\cos\theta} + \frac{1}{\cos^2\theta} + \cos^2\theta + \frac{2\cos\theta}{\sin\theta} + \frac{1}{\sin^2\theta}$$

$$= (\sin^2\theta + \cos^2\theta) + \left(\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}\right) + \left(\frac{2\sin\theta}{\cos\theta} + \frac{2\cos\theta}{\sin\theta}\right)$$

$$= 1 + \frac{1}{\sin^2\theta \,.\, \cos^2\theta} + \frac{2}{\sin\theta \,.\, \cos\theta} = \left(1 + \frac{1}{\sin\theta . \cos\theta}\right)^2 = (1 + \sec\theta . \operatorname{cosec}\theta)^2.$$

(24) $\dfrac{1 + (\operatorname{cosec}\theta \,.\, \tan\phi)^2}{1 + (\operatorname{cosec}a \,.\, \tan\phi)^2} = \dfrac{1 + \dfrac{\sin^2\phi}{\sin^2\theta \,.\, \cos^2\phi}}{1 + \dfrac{\sin^2\phi}{\sin^2 a \,.\, \cos^2\phi}} = \dfrac{\sin^2\theta \,.\, \cos^2\phi + \sin^2\phi}{\sin^2 a \,.\, \cos^2\phi + \sin^2\phi} \cdot \dfrac{\sin^2 a}{\sin^2\theta}$

$$= \frac{\sin^2\theta(1 - \sin^2\phi) + \sin^2\phi}{\sin^2 a(1 - \sin^2\phi) + \sin^2\phi} \cdot \frac{\sin^2 a}{\sin^2\theta} = \frac{\sin^2\theta - \sin^2\theta \,.\, \sin^2\phi + \sin^2\phi}{\sin^2 a - \sin^2 a \,.\, \sin^2\phi + \sin^2\phi} \cdot \frac{\sin^2 a}{\sin^2\theta}$$

$$= \frac{\sin^2\theta + \sin^2\phi \,.\, \cos^2\theta}{\sin^2 a + \sin^2\phi \,.\, \cos^2 a} \cdot \frac{\sin^2 a}{\sin^2\theta}$$

$$= \frac{1 + \sin^2\phi \,.\, \cot^2\theta}{1 + \sin^2\phi \,.\, \cot^2 a} = \frac{1 + (\cot\theta \,.\, \sin\phi)^2}{1 + (\cot a \,.\, \sin\phi)^2}.$$

(25) $(3 - 4\sin^2 A)(1 - 3\tan^2 A) = (3 - 4\sin^2 A)\left(1 - \dfrac{3\sin^2 A}{\cos^2 A}\right)$

$$= (3 - 4\sin^2 A)\left(\frac{\cos^2 A - 3\sin^2 A}{\cos^2 A}\right)$$

$$= (3 - 4\sin^2 A)\left(\frac{\cos^2 A - 3(1 - \cos^2 A)}{\cos^2 A}\right)$$

$$= \frac{3 - 4\sin^2 A}{\cos^2 A} \cdot (4\cos^2 A - 3)$$

$$= \frac{3\cos^2 A + 3\sin^2 A - 4\sin^2 A}{\cos^2 A}(4\cos^2 A - 3)$$

$$= \frac{3\cos^2 A - \sin^2 A}{\cos^2 A}(4\cos^2 A - 3)$$

$$= (3 - \tan^2 A)(4\cos^2 A - 3).$$

EXAMPLES—XX. (p. 65).

1. (1) $90^\circ-(24^\circ.\,14'.\,42'')=65^\circ.\,45'.\,18''.$

(2) $90^\circ-(43^\circ.\,2'.\,57'')=46^\circ.\,57'.\,3''.$

(3) $90^\circ-(64^\circ.\,0'.\,14'')=25^\circ.\,59'.\,46''.$

(4) $90^\circ-(82^\circ.\,4'.\,15'')=7^\circ.\,55'.\,45''.$

(5) $90^\circ-(125^\circ.\,15'.\,42'')=-(35^\circ.\,15'.\,42'').$

(6) $90^\circ-(178^\circ.\,27'.\,34'')=-(88^\circ.\,27'.\,34'').$

(7) $90^\circ-195^\circ=-105^\circ.$

(8) $90^\circ-254^\circ=-164^\circ.$

(9) $90^\circ-(-25^\circ)=90^\circ+25^\circ=115^\circ.$

(10) $90^\circ-(-245^\circ)=90^\circ+245^\circ=335^\circ.$

2. (1) $100^g-(32^g.\,23^`.\,24^{``})=67^g.\,76^`.\,76^{``}.$

(2) $100^g-(95^g.\,3^`.\,75^{``})=4^g.\,96^`.\,25^{``}.$

(3) $100^g-(46^g.\,0^`.\,84^{``})=53^g.\,99^`.\,16^{``}.$

(4) $100^g-(2^g.\,5^`.\,4^{``})=97^g.\,94^`.\,96^{``}.$

(5) $100^g-(135^g.\,2^`.\,5^{``})=-(35^g.\,2^`.\,5^{``}).$

(6) $100^g-(169^g.\,0^`.\,3^{``})=-(69^g.\,0^`.\,3^{``}).$

(7) $100^g-243^g=-143^g.$

(8) $100^g-357^g=-257^g.$

(9) $100^g-(-35^g)=100^g+35^g=135^g.$

(10) $100^g-(-245^g)=100^g+245^g=345^g.$

3. (1) $\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}.$ (2) $\frac{\pi}{2}-\frac{\pi}{3}=\frac{\pi}{6}.$ (3) $\frac{\pi}{2}-\frac{3\pi}{5}=-\frac{\pi}{10}.$

(4) $\frac{\pi}{2}-\left(-\frac{\pi}{4}\right)=\frac{\pi}{2}+\frac{\pi}{4}=\frac{3\pi}{4}.$ (5) $\frac{\pi}{2}-\left(-\frac{3\pi}{4}\right)=\frac{\pi}{2}+\frac{3\pi}{4}=\frac{5\pi}{4}.$

Examples—XXI. (p. 68).

1. (1) $180^\circ - (34^\circ.\,12'.\,49'') = 145^\circ.\,47'.\,11''$.

(2) $180^\circ - (132^\circ.\,24'.\,47'') = 47^\circ.\,35'.\,13''$.

(3) $180^\circ - (146^\circ.\,0'.\,41'') = 33^\circ.\,59'.\,19''$.

(4) $180^\circ - (28^\circ.\,15'.\,4'') = 151^\circ.\,44'.\,56''$.

(5) $180^\circ - (179^\circ.\,59'.\,59'') = \quad 1''$.

(6) $180^\circ - (100^\circ.\,49'.\,53'') = 79^\circ.\,10'.\,7''$.

(7) $180^\circ - 245^\circ \quad = -65^\circ$.

(8) $180^\circ - (437^\circ.\,3'.\,4'') = -(257^\circ.\,3'.\,4'')$.

(9) $180^\circ - (-49^\circ) = 180^\circ + 49^\circ = 229^\circ$.

(10) $180^\circ - (-355^\circ) = 180^\circ + 355^\circ = 535^\circ$.

2. (1) $200^g - (132^g.\,32'.\,42'') = 67^g.\,67'.\,58''$.

(2) $200^g - (195^g.\,2'.\,57'') = 4^g.\,97'.\,43''$.

(3) $200^g - (3^g.\,97'.\,98'') = 196^g.\,2'.\,2''$.

(4) $200^g - (65^g.\,12'.\,8'') = 134^g.\,87'.\,92''$.

(5) $200^g - (154^g.\,3'.\,6'') = 45^g.\,96'.\,94''$.

(6) $200^g - (174^g.\,0'.\,4'') = 25^g.\,99'.\,96''$.

(7) $200^g - 275^g = -75^g$.

(8) $200^g - (527^g.\,2'.\,14'') = \quad (327^g.\,2'.\,14'')$.

(9) $200^g - (-35^g) = 200^g + 35^g = 235^g$.

(10) $200^g - (-325^g) = 200^g + 325^g = 525^g$.

3. (1) $\pi - \dfrac{\pi}{2} = \dfrac{\pi}{2}$. (2) $\pi - \dfrac{\pi}{3} = \dfrac{2\pi}{3}$. (3) $\pi - \dfrac{4\pi}{5} = \dfrac{\pi}{5}$.

(4) $\pi - \left(-\dfrac{\pi}{4}\right) = \pi + \dfrac{\pi}{4} = \dfrac{5\pi}{4}$. (5) $\pi - \left(-\dfrac{3\pi}{4}\right) = \pi + \dfrac{3\pi}{4} = \dfrac{7\pi}{4}$.

4. Let θ be the circular measure of the angle.

Then $\frac{\pi}{2}-\theta$ is the complement of θ;

and $\pi-\left(\frac{\pi}{2}-\theta\right)$, or, $\frac{\pi}{2}+\theta$ is the supplement of the complement of θ.

Again $\pi-\theta$ is the supplement of θ,

and $\frac{\pi}{2}-(\pi-\theta)$, or, $\theta-\frac{\pi}{2}$ is the complement of the supplement of θ;

$$\therefore \text{ difference}=\frac{\pi}{2}+\theta-\left(\theta-\frac{\pi}{2}\right)=\frac{\pi}{2}+\frac{\pi}{2}=\pi.$$

Examples—XXII. (p. 72).

1. (1) Take the construction and notation of Art. 101.

$$\text{Then } \sec(180^\circ-A)=\sec EOP'=\frac{OP'}{OM'}=\frac{OP}{-OM}=-\sec A.$$

(2) Take the construction of Art. 102, and let $\angle\, EOP=\theta$.

$$\text{Then } \operatorname{cosec}\left(\frac{\pi}{2}+\theta\right)=\frac{OP'}{P'M'}=\frac{OP}{OM}=\sec\theta.$$

(3) Take the construction and notation of Art. 103.

$$\text{Then } \tan(180^\circ+A)=\frac{P'M'}{OM'}=\frac{-PM}{-OM}=\frac{PM}{OM}=\tan A.$$

(4) Take the construction of Art. 103, and let $\angle\, EOP=\theta$.

$$\text{Then } \sec(\pi+\theta)=\frac{OP'}{OM'}=\frac{OP}{-OM}=-\sec\theta.$$

(5) Take the construction of Art. 104, and let $\angle\, EOP=\theta$.

$$\text{Then } \tan(-\theta)=\frac{MP'}{MO}=\frac{-MP}{MO}=-\tan\theta.$$

(6) Take the construction of Art. 104, and let $\angle\, EOP=\theta$.

$$\text{Then } \cot(2\pi-\theta)=\cot EOP'=\frac{OM}{MP'}=\frac{OM}{-MP}=-\cot\theta.$$

2. (1) Take the construction of Art. 102, and let $\angle EOP = B$.

Then $\operatorname{cosec}(90° + B) = \operatorname{cosec} EOP' = \dfrac{OP'}{P'M'} = \dfrac{OP}{OM} = \sec B = \dfrac{\operatorname{cosec} B}{\sqrt{\operatorname{cosec}^2 B - 1}}$.

(Ex. XVII. 2.)

(2) Take the construction of Art. 103, and let $\angle EOP = \phi$.

Then $\operatorname{cosec}(\pi + \phi) = \operatorname{cosec} EOP' = \dfrac{OP'}{P'M'} = \dfrac{OP}{-PM} = -\operatorname{cosec}\phi$.

3. (1) Take the construction of Art. 102, and let $\angle EOP = A$.

Then $\sec(90° + A) = \sec EOP' = \dfrac{OP'}{OM'} = \dfrac{OP}{-PM} = -\operatorname{cosec} A = -\dfrac{\sec A}{\sqrt{\sec^2 A - 1}}$.

(Ex. XVII. 3.)

(2) Take the construction of Art. 99, and let $\angle EOP = \theta$.

Then $\sec\left(\dfrac{\pi}{2} - \theta\right) = \sec EOP' = \dfrac{OP'}{OM'} = \dfrac{OP}{PM} = \operatorname{cosec}\theta = \dfrac{\sec\theta}{\sqrt{\sec^2\theta - 1}}$.

(Ex. XVII. 3.)

EXAMPLES—XXIII. (p. 72).

(1) $\sin 120° = \sin(180° - 120°) = \sin 60° = \dfrac{\sqrt{3}}{2}$.

(2) $\cos 120° = -\cos(180° - 120°) = -\cos 60° = -\dfrac{1}{2}$.

(3) $\sin 135° = \sin(180° - 135°) = \sin 45° = \dfrac{1}{\sqrt{2}}$.

(4) $\cos 135° = -\cos(180° - 135°) = -\cos 45° = -\dfrac{1}{\sqrt{2}}$.

(5) $\sin 150° = \sin(180° - 150°) = \sin 30° = \dfrac{1}{2}$.

(6) $\cos 150° = -\cos(180° - 150°) = -\cos 30° = -\dfrac{\sqrt{3}}{2}$.

(7) $\sin 225° = \sin(180° + 45°) = -\sin 45° = -\dfrac{1}{\sqrt{2}}$.

(8) $\sin 240° = \sin(180° + 60°) = -\sin 60° = -\dfrac{\sqrt{3}}{2}$.

(9) $\tan 300° = \tan(360° - 60°) = -\tan 60° = -\sqrt{3}$.

(10) $\operatorname{cosec} 300° = \operatorname{cosec}(360° - 60°) = -\operatorname{cosec} 60° = -\frac{2}{\sqrt{3}}$.

(11) $\sec 315° = \sec(360° - 45°) = \sec 45° = \sqrt{2}$.

(12) $\cot 330° = \cot(360° - 30°) = -\cot 30° = -\sqrt{3}$.

EXAMPLES—XXIV. (p. 75).

(1) $\sin\theta + \cos\theta = 0$,

$$\sin\theta = -\cos\theta,$$
$$\sin^2\theta = \cos^2\theta,$$
$$\sin^2\theta = 1 - \sin^2\theta,$$
$$2\sin^2\theta = 1.$$

Hence $\sin\theta = \pm\frac{1}{\sqrt{2}}$, and $\therefore\ \theta = 45°$ or $-45°$.

The latter of these values must be taken, because $\sin\theta$ and $\cos\theta$ must have different signs to satisfy the equation.

(2) $\sin\theta - \cos\theta = 0$,

$$\sin\theta = \cos\theta,$$

and, as in Example (1), $\theta = 45°$ or $-45°$.

The former of these values must be taken, because $\sin\theta$ and $\cos\theta$ must have the same sign to satisfy the equation.

(3) $\sin\theta = \tan\theta$,

$\sin\theta = \frac{\sin\theta}{\cos\theta}$, and, dividing by $\sin\theta$,

$1 = \frac{1}{\cos\theta}$, or, $\cos\theta = 1$, and $\therefore\ \theta = 0°$.

(4) $\cos\theta = \cot\theta$,

$\cos\theta = \frac{\cos\theta}{\sin\theta}$, and, dividing by $\cos\theta$,

$1 = \frac{1}{\sin\theta}$, or, $\sin\theta = 1$, and $\therefore\ \theta = 90°$.

(5) $2\sin\theta=\tan\theta$,

$$2\sin\theta=\frac{\sin\theta}{\cos\theta}, \text{ or, } 2\cos\theta=1, \text{ or, } \cos\theta=\frac{1}{2}, \text{ and, } \therefore, \theta=60°.$$

Also, since we divided by $\sin\theta$, one value of θ to satisfy the original equation is given by $\sin\theta=0$, or, $\theta=0°$.

(6)
$$3\sin\theta=2\cos^2\theta,$$
$$3\sin\theta=2(1-\sin^2\theta),$$
$$2\sin^2\theta+3\sin\theta=2,$$
$$\sin^2\theta+\frac{3}{2}\sin\theta=1.$$
$$\sin^2\theta+\frac{3}{2}\sin\theta+\frac{9}{16}=\frac{25}{16}.$$
$$\sin\theta+\frac{3}{4}=\pm\frac{5}{4}.$$
$$\text{Hence } \sin\theta=\frac{1}{2}, \text{ or, } -2.$$

The second value is inadmissible

$$\therefore\ \sin\theta=\frac{1}{2}, \text{ or, } \theta=30°.$$

(7)
$$\sin\theta+\cos^2\theta\,.\,\operatorname{cosec}\theta=2,$$
$$\sin\theta+\frac{\cos^2\theta}{\sin\theta}=2,$$
$$\sin^2\theta+\cos^2\theta=2\sin\theta,$$
$$1=2\sin\theta.$$
$$\text{Hence } \sin\theta=\frac{1}{2}, \text{ or, } \theta=30°.$$

(8)
$$\tan\theta=4-3\cot\theta,$$
$$\tan\theta+3\cot\theta=4,$$
$$\tan\theta+\frac{3}{\tan\theta}=4,$$
$$\tan^2\theta+3=4\tan\theta,$$
$$\tan^2\theta-4\tan\theta=-3,$$
$$\tan^2\theta-4\tan\theta+4=1,$$
$$\tan\theta-2=\pm1.$$

Hence $\tan\theta=3$ or 1, and the latter of these values of $\tan\theta$ enables us to say that *one* value of θ is $45°$.

(9) $$4\sec^2\theta - 7\tan^2\theta = 3,$$
$$4(1+\tan^2\theta) - 7\tan^2\theta = 3,$$
$$4 - 3\tan^2\theta = 3,$$
$$\tan^2\theta = \frac{1}{3}, \text{ or, } \tan\theta = \frac{1}{\sqrt{3}}, \text{ and } \therefore \theta = 30°.$$

(10) $$\cos\theta . \operatorname{cosec}\theta + \sin\theta . \sec\theta = \frac{4}{\sqrt{3}},$$
$$\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} = \frac{4}{\sqrt{3}},$$
$$\frac{\cos^2\theta + \sin^2\theta}{\sin\theta . \cos\theta} = \frac{4}{\sqrt{3}},$$
$$\sqrt{3} = 4\sin\theta . \cos\theta,$$
$$3 = 16\sin^2\theta(1 - \sin^2\theta),$$
$$16\sin^4\theta - 16\sin^2\theta = -3,$$
$$\sin^4\theta - \sin^2\theta = -\frac{3}{16}.$$
Hence $\sin^2\theta = \frac{3}{4}$ or $\frac{1}{4}$,

and $\therefore \sin\theta = \frac{\sqrt{3}}{2}$ or $\frac{1}{2}$, and $\theta = 60°$ or $30°$.

(11) $$3\sin^2\theta - \cos^2\theta + (\sqrt{3}+1)(1 - 2\sin\theta) = 0,$$
$$3\sin^2\theta - (1 - \sin^2\theta) + \sqrt{3} + 1 - 2\sqrt{3}\sin\theta - 2\sin\theta = 0,$$
$$4\sin^2\theta - 2(\sqrt{3}+1)\sin\theta = -\sqrt{3},$$
$$\sin^2\theta - \frac{\sqrt{3}+1}{2} \cdot \sin\theta = -\frac{\sqrt{3}}{4},$$
$$\sin^2\theta - \frac{\sqrt{3}+1}{2}\sin\theta + \frac{4+2\sqrt{3}}{16} = \frac{4+2\sqrt{3}}{16} - \frac{\sqrt{3}}{4} = \frac{4-2\sqrt{3}}{16},$$
$$\sin\theta - \frac{\sqrt{3}+1}{4} = \pm\frac{\sqrt{3}-1}{4}.$$
Hence $\sin\theta = \frac{\sqrt{3}}{2}$ or $\frac{1}{2}$, and $\theta = 60°$ or $30°$.

(12) $$3\cos^2\theta - \sin^2\theta + (\sqrt{3}+1)(1 - 2\cos\theta) = 0,$$
$$3\cos^2\theta - (1 - \cos^2\theta) + \sqrt{3} + 1 - 2\sqrt{3}\cos\theta - 2\cos\theta = 0,$$
$$4\cos^2\theta - 2(\sqrt{3}+1)\cos\theta = -\sqrt{3}.$$
Hence, by the same process as in Example (11),
$$\cos\theta = \frac{\sqrt{3}}{2} \text{ or } \frac{1}{2}, \text{ and } \theta = 30° \text{ or } 60°.$$

(13)
$$\sec\theta \, . \, \mathrm{cosec}\theta + 2\cot\theta = 4,$$
$$\frac{1}{\cos\theta \, . \, \sin\theta} + \frac{2\cos\theta}{\sin\theta} = 4,$$
$$1 + 2\cos^2\theta = 4\sin\theta \, . \, \cos\theta,$$
$$1 + 4\cos^2\theta + 4\cos^4\theta = 16\sin^2\theta \, . \, \cos^2\theta,$$
$$1 + 4\cos^2\theta + 4\cos^4\theta = 16\cos^2\theta - 16\cos^4\theta,$$
$$20\cos^4\theta - 12\cos^2\theta = -1,$$
$$\cos^4\theta - \frac{3}{5}\cos^2\theta = -\frac{1}{20}.$$

Hence $\cos^2\theta = \frac{1}{2}$, and $\cos\theta = \frac{1}{\sqrt{2}}$, and $\theta = 45°$

(14)
$$\sin\theta + \cos\theta = \sqrt{2}, \quad . \quad . \quad . \quad . \quad (1),$$
$$\sin^2\theta + 2\sin\theta \, . \, \cos\theta + \cos^2\theta = 2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$
$$2\sin\theta \, . \, \cos\theta = 1,$$

$4\sin\theta \, . \, \cos\theta = 2$, and, subtracting this from (2),
$$\sin^2\theta - 2\sin\theta \, . \, \cos\theta + \cos^2\theta = 0,$$

$\sin\theta - \cos\theta = 0$, and, adding this to (1),

$2\sin\theta = \sqrt{2}$, $\therefore$ $\sin\theta = \frac{1}{\sqrt{2}}$, and $\theta = 45°$.

(15)
$$\cot^2\theta + 4\cos^2\theta = 6,$$
$$\cos^2\theta + 4\cos^2\theta \, . \, \sin^2\theta = 6\sin^2\theta,$$
$$\cos^2\theta + 4\cos^2\theta - 4\cos^4\theta = 6 - 6\cos^2\theta,$$
$$\cos^4\theta - \frac{11}{4}\cos^2\theta = -\frac{3}{2}.$$

Hence $\cos^2\theta = \frac{3}{4}$ or 2.

The latter value is inadmissible, and we must have

$\cos^2\theta = \frac{3}{4}$, or, $\cos\theta = \frac{\sqrt{3}}{2}$, and $\theta = 30°$.

(16)
$$\tan\theta + \cot\theta = 2.$$
$$\tan\theta + \frac{1}{\tan\theta} = 2,$$
$$\tan^2\theta - 2\tan\theta = -1;$$

$\therefore$ $\tan\theta = 1$, and $\theta = 45°$.

(17) $$\sin\theta - \cos\theta = \sqrt{2}, \quad . \quad . \quad . \quad . \quad (1)$$

$$\sin^2\theta - 2\sin\theta . \cos\theta + \cos^2\theta = 2, \quad . \quad . \quad . \quad . \quad (2)$$

$$-2\sin\theta . \cos\theta = 1,$$

$$4\sin\theta . \cos\theta = -2, \text{ and adding this to (2)}$$

$$\sin^2\theta + 2\sin\theta . \cos\theta + \cos^2\theta = 0,$$

$$\sin\theta + \cos\theta = 0, \text{ and adding this to (1)}$$

$$2\sin\theta = \sqrt{2}, \text{or, } \sin\theta = \frac{1}{\sqrt{2}}, \text{ and } \theta = 45° \text{ or } 135°.$$

Now $\cos\theta$ has to be of the same numerical value as $\sin\theta$, but with a different sign, and hence $45°$ is an inadmissible value of θ;

$$\therefore \theta = 135°.$$

(18) $$\sin\theta + \cos\theta = 2\sqrt{2} . \sin\theta . \cos\theta,$$

$$\sin^2\theta + 2\sin\theta . \cos\theta + \cos^2\theta = 8\sin^2\theta . \cos^2\theta,$$

$$8\sin^2\theta . \cos^2\theta - 2\sin\theta . \cos\theta = 1,$$

$$\sin^2\theta . \cos^2\theta - \frac{1}{4} . \sin\theta . \cos\theta = \frac{1}{8},$$

$$\sin^2\theta . \cos^2\theta - \frac{1}{4}\sin\theta . \cos\theta + \frac{1}{64} = \frac{9}{64};$$

$$\therefore \sin\theta . \cos\theta = \frac{1}{2} \text{ or } -\frac{1}{4}.$$

Taking the former of these values, we get

$$\sin^2\theta\,(1 - \sin^2\theta) = \frac{1}{4}.$$

Whence $\sin^2\theta = \frac{1}{2}$, or, $\sin\theta = \frac{1}{\sqrt{2}}$, and $\theta = 45°$.

(19) $$\sqrt{3} . \sin\theta = \sqrt{3} - \cos\theta,$$

$$3\sin^2\theta = 3 - 2\sqrt{3} . \cos\theta + \cos^2\theta,$$

$$3 - 3\cos^2\theta = 3 - 2\sqrt{3}\cos\theta + \cos^2\theta,$$

$$4\cos^2\theta = 2\sqrt{3} . \cos\theta.$$

Dividing by $\cos\theta$, we get

$$4\cos\theta = 2\sqrt{3}, \text{ or, } \cos\theta = 0.$$

Hence $\cos\theta = \frac{\sqrt{3}}{2}$, or, $\cos\theta = 0$;

$$\therefore \theta = 30° \text{ or } 90°.$$

(20)
$$\tan^2\theta + 4\sin^2\theta = 3,$$
$$\sin^2\theta + 4\sin^2\theta \,.\, \cos^2\theta = 3\cos^2\theta,$$
$$1 - \cos^2\theta + 4\cos^2\theta - 4\cos^4\theta = 3\cos^2\theta,$$
$$4\cos^4\theta = 1\ ;$$
$$\therefore \text{ one value of } \cos\theta \text{ is } \frac{1}{\sqrt{2}}, \text{ or, } \theta = 45^\circ.$$

EXAMPLES—XXV. (p. 81).

(1) $\sin 480^\circ = \sin(360^\circ + 120^\circ) = \sin 120^\circ = \sin 60^\circ = \frac{\sqrt{3}}{2}.$

(2) $\cos 480^\circ = \cos(360^\circ + 120^\circ) = \cos 120^\circ = -\cos 60^\circ = -\frac{1}{2}.$

(3) $\sin 495^\circ = \sin(360^\circ + 135^\circ) = \sin 135^\circ = \sin 45^\circ = \frac{1}{\sqrt{2}}.$

(4) $\cos 495^\circ = \cos(360^\circ + 135^\circ) = \cos 135^\circ = -\cos 45^\circ = -\frac{1}{\sqrt{2}}.$

(5) $\sin 870^\circ = \sin(720^\circ + 150^\circ) = \sin 150^\circ = \sin 30^\circ = \frac{1}{2}.$

(6) $\cos 870^\circ = \cos(720^\circ + 150^\circ) = \cos 150^\circ = -\cos 30^\circ = -\frac{\sqrt{3}}{2}.$

(7) $\sin 945^\circ = \sin(720^\circ + 225^\circ) = \sin 225^\circ = -\sin 45^\circ = -\frac{1}{\sqrt{2}}.$

(8) $\sin 960^\circ = \sin(720^\circ + 240^\circ) = \sin 240^\circ = -\sin 60^\circ = -\frac{\sqrt{3}}{2}.$

(9) $\tan 1020^\circ = \tan(720^\circ + 300^\circ) = \tan 300^\circ = -\tan 60^\circ = -\sqrt{3}.$

(10) $\operatorname{cosec} 1380^\circ = \operatorname{cosec}(1080^\circ + 300^\circ) = \operatorname{cosec} 300^\circ = -\operatorname{cosec} 60^\circ = -\frac{2}{\sqrt{3}}.$

(11) $\sec 1395^\circ = \sec(1080^\circ + 315^\circ) = \sec 315^\circ = \sec 45^\circ = \sqrt{2}.$

(12) $\cot 1410^\circ = \cot(1080^\circ + 330^\circ) = \cot 330^\circ = -\cot 30^\circ = -\sqrt{3}.$

(13) $\cos 420° = \cos(360° + 60°) = \cos 60° = \frac{1}{2}$.

(14) $\sec 750° = \sec(720° + 30°) = \sec 30° = \frac{2}{\sqrt{3}}$.

(15) $\tan 945° = \tan(720° + 225°) = \tan 225° = \tan 45° = 1$.

(16) $\sin 1200° = \sin(1080° + 120°) = \sin 120° = \sin 60° = \frac{\sqrt{3}}{2}$.

(17) $\sin 1485° = \sin(1440° + 45°) = \sin 45° = \frac{1}{\sqrt{2}}$.

(18) $\cos 1470° = \cos(1440° + 30°) = \cos 30° = \frac{\sqrt{3}}{2}$.

(19) $\sin 7\pi = \sin(6\pi + \pi) = \sin\pi = 0$.

(20) $\sec 8\pi = \sec 2\pi = 1$.

(21) $\operatorname{cosec} 930° = \operatorname{cosec}(720° + 210°) = \operatorname{cosec} 210° = -\operatorname{cosec} 30° = -2$.

(22) $\cot 1140° = \cot(1080° + 60°) = \cot 60° = \frac{1}{\sqrt{3}}$.

(23) $\tan 1305° = \tan(1080° + 225°) = \tan 225° = \tan 45° = 1$.

(24) $\operatorname{cosec} 1740° = \operatorname{cosec}(1440° + 300°) = \operatorname{cosec} 300° = -\operatorname{cosec} 60° = -\frac{2}{\sqrt{3}}$.

(25) $\sin(-240°) = -\sin 240° = -\sin(-60°) = \sin 60° = \frac{\sqrt{3}}{2}$.

(26) $\cot(-675°) = \cot(720° - 675°) = \cot 45° = 1$.

(27) $\sec(-135°) = -\sec(180° - 135°) = -\sec 45° = -\sqrt{2}$.

(28) $\tan(-225°) = \tan(360° - 225°) = \tan 135° = -\tan 45° = -1$.

(29) $\operatorname{cosec}(-690°) = \operatorname{cosec}(720° - 690°) = \operatorname{cosec} 30° = 2$.

(30) $\cos(-120°) = \cos 120° = -\cos 60° = -\frac{1}{2}$.

EXAMPLES—XXVI. (p. 82).

(1) $\sin\theta=1$; $\therefore$ one value of θ is $\frac{\pi}{2}$;

$\therefore$ general value of θ is $n\pi+(-1)^n.\frac{\pi}{2}$.

(2) $\cos\theta=1$; $\therefore$ one value of θ is 0 ;

$\therefore$ general value of θ is $2n\pi$.

(3) $\sin\theta=\frac{1}{\sqrt{2}}$; $\therefore$ one value of θ is $\frac{\pi}{4}$;

$\therefore$ general value of θ is $n\pi+(-1)^n.\frac{\pi}{4}$.

(4) $\tan\theta=\sqrt{3}$; $\therefore$ one value of θ is $\frac{\pi}{3}$;

$\therefore$ general value of θ is $n\pi+\frac{\pi}{3}$.

(5)
$$3\sin\theta=2\cos^2\theta$$
$$3\sin\theta=2(1-\sin^2\theta)$$
$$\sin^2\theta+\frac{3}{2}\sin\theta=1$$
$$\left(\sin\theta+\frac{3}{4}\right)^2=\pm\frac{5}{4},\text{ or, }\sin\theta=\frac{1}{2}\text{ or }-2$$

$\therefore$ least positive value of θ is $\frac{\pi}{6}$;

$\therefore$ general value of θ is $n\pi+(-1)^n.\frac{\pi}{6}$.

(6) $2\sin\theta=\tan\theta$,

$$2\sin\theta=\frac{\sin\theta}{\cos\theta};$$

$\therefore \sin\theta=0$, or, $\cos\theta=\frac{1}{2}$;

$\therefore \theta=0$, or, $\theta=\frac{\pi}{3}$;

$\therefore$ general value of θ is $n\pi$ or $2n\pi\pm\frac{\pi}{3}$.

(7) $$\tan^2\theta + 4\sin^2\theta = 3,$$

$$\sin^2\theta + 4\sin^2\theta \cdot \cos^2\theta = 3\cos^2\theta,$$

$$\sin^2\theta + 4\sin^2\theta - 4\sin^4\theta = 3 - 3\sin^2\theta,$$

$$4\sin^4\theta - 8\sin^2\theta = -3,$$

$$\sin^4\theta - 2\sin^2\theta + 1 = \frac{1}{4},$$

$$\sin^2\theta - 1 = \pm\frac{1}{2}.$$

Hence $\sin\theta = \pm\sqrt{\frac{3}{2}}$ or $\pm\frac{1}{\sqrt{2}}$;

$\therefore$ least positive value of θ is $\frac{\pi}{4}$;

$\therefore$ general value of θ is $n\pi + (-1)^n \cdot \frac{\pi}{4}$.

(8) $$\cos^2 = \sin^2\theta,$$

$$\cos^2\theta = 1 - \cos^2\theta,$$

$2\cos^2\theta = 1$, and $\therefore \cos\theta = \pm\frac{1}{\sqrt{2}}$;

$\therefore$ the least positive values of θ are $\frac{\pi}{4}$ and $\frac{3\pi}{4}$;

$\therefore$ the general value of θ is $2n\pi \pm \frac{\pi}{4}$ or $2n\pi \pm \frac{3\pi}{4}$.

(9) $$\tan\theta = 4 - 3\cot\theta,$$

$$\tan\theta + \frac{3}{\tan\theta} = 4,$$

$$\tan^2\theta - 4\tan\theta = -3,$$

$$\tan\theta = 3 \text{ or } 1;$$

$\therefore$ the least positive value of θ is $\frac{\pi}{4}$;

$\therefore$ general value of θ is $n\pi + \frac{\pi}{4}$.

$$(10)\ \sec^2\theta - \frac{5}{2}\sec\theta + 1 = 0,$$

$$\sec^2\theta - \frac{5}{2}\sec\theta + \frac{25}{16} = \frac{9}{16},$$

$$\sec\theta - \frac{5}{4} = \pm\frac{3}{4};$$

$$\therefore \sec\theta = 2 \text{ or } \frac{1}{2}.$$

Taking the value 2 for $\sec\theta$ (the other value being impossible) the general value of θ is $2n\pi \pm \frac{\pi}{3}$.

EXAMPLES—XXVII. (p. 87).

(1)

$$\begin{aligned}\sin(A+B).\sin(A-B) &= (\sin A.\cos B + \cos A.\sin B).(\sin A.\cos B - \cos A.\sin B)\\ &= \sin^2 A.\cos^2 B - \cos^2 A.\sin^2 B\\ &= \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B\\ &= \sin^2 A - \sin^2 B.\end{aligned}$$

(2)

$$\begin{aligned}\sin(\alpha+\beta).\sin(\alpha-\beta) &= \sin\alpha.\cos\beta + \cos\alpha.\sin\beta)(\sin\alpha.\cos\beta - \cos\alpha.\sin\beta)\\ &= \sin^2\alpha.\cos^2\beta - \cos^2\alpha.\sin^2\beta\\ &= (1-\cos^2\alpha)\cos^2\beta - \cos^2\alpha(1-\cos^2\beta)\\ &= \cos^2\beta - \cos^2\alpha.\end{aligned}$$

(3)

$$\begin{aligned}\cos(A+B).\cos(A-B) &= (\cos A.\cos B - \sin A.\sin B)(\cos B.\cos B + \sin A.\sin B)\\ &= \cos^2 A.\cos^2 B - \sin^2 A.\sin^2 B\\ &= \cos^2 A(1-\sin^2 B) - (1-\cos^2 A)\sin^2 B\\ &= \cos^2 A - \sin^2 B.\end{aligned}$$

(4)

$$\begin{aligned}\cos(\alpha+\beta).\cos(\alpha-\beta) &= (\cos\alpha.\cos\beta - \sin\alpha.\sin\beta)(\cos\alpha.\cos\beta + \sin\alpha.\sin\beta)\\ &= \cos^2\alpha.\cos^2\beta - \sin^2\alpha.\sin^2\beta\\ &= (1-\sin^2\alpha)\cos^2\beta - \sin^2\alpha(1-\cos^2\beta)\\ &= \cos^2\beta - \sin^2\alpha.\end{aligned}$$

(5)

$$2\sin(x+y).\cos(x-y)=2(\sin x.\cos y+\cos x.\sin y).(\cos x.\cos y+\sin x.\sin y)$$
$$=2\{\sin x.\cos x.\cos^2 y+\sin^2 x.\cos y.\sin y+\cos^2 x.\sin y.\cos y+\sin x.\cos x.\sin^2 y\}$$
$$=2\{\sin x.\cos x(\cos^2 y+\sin^2 y)+\sin y.\cos y(\sin^2 x+\cos^2 x)\}$$
$$=2\{\sin x.\cos x+\sin y.\cos y\}$$
$$=(\sin x.\cos x+\cos x.\sin x)+(\sin y.\cos y+\cos y.\sin y)$$
$$=\sin(x+x)+\sin(y+y)$$
$$=\sin 2x+\sin 2y$$

(6)

$$2\cos(x+y).\sin(x-y)=2(\cos x.\cos y-\sin x.\sin y).(\sin x.\cos y-\cos x.\sin y)$$
$$=2\{\sin x.\cos x.\cos^2 y-\sin y.\cos y.\cos^2 x-\sin y.\cos y.\sin^2 x+\sin x.\cos x.\sin^2 y\}$$
$$=2\{\sin x.\cos x.(\cos^2 y+\sin^2 y)-\sin y.\cos y.(\cos^2 x+\sin^2 x)\}$$
$$=2\{\sin x.\cos x-\sin y.\cos y\}$$
$$=(\sin x.\cos x+\cos x.\sin x)-(\sin y.\cos y+\cos y.\sin y)$$
$$=\sin 2x-\sin 2y.$$

(7) $$\tan A+\tan B=\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}.$$
$$=\frac{\sin A.\cos B+\cos A.\sin B}{\cos A.\cos B}$$
$$=\frac{\sin(A+B)}{\cos A.\cos B}.$$

(8) $$\tan\alpha-\tan\beta=\frac{\sin\alpha}{\cos\alpha}-\frac{\sin\beta}{\cos\beta}$$
$$=\frac{\sin\alpha.\cos\beta-\cos\alpha.\sin\beta}{\cos\alpha.\cos\beta}$$
$$=\frac{\sin(\alpha-\beta)}{\cos\alpha.\cos\beta}.$$

EXAMPLES—XXVIII. (p. 88).

(1) $\sin 15° = \sin(45° - 30°)$

$$= \sin 45° . \cos 30° - \cos 45° . \sin 30°$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

(2) $\cos 75° = \cos(45° + 30°)$

$$= \cos 45° . \cos 30° - \sin 45° . \sin 30°$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

(3) $\tan 75° = \tan(45° + 30°)$

$$= \sin(45° + 30°) \div \cos(45° + 30°)$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}} \div \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{(\sqrt{3}+1)^2}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{4+2\sqrt{3}}{3-1} = 2 + \sqrt{3}.$$

(4) $\cot 75° = \cos 75° \div \sin 75°$

$$= \cos(45° + 30°) \div \sin(45° + 30°)$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}} \div \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$= \frac{(\sqrt{3}-1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{4-2\sqrt{3}}{3-1} = 2 - \sqrt{3}.$$

(5) If $\sin\alpha = \frac{1}{3}$, $\cos\alpha = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$.

If $\sin\beta = \frac{2}{3}$, $\cos\beta = \frac{\sqrt{5}}{3}$;

$$\therefore \sin(\alpha + \beta) = \frac{1}{3} \cdot \frac{\sqrt{5}}{3} + \frac{2\sqrt{2}}{3} \cdot \frac{2}{3} = \frac{\sqrt{5}+4\sqrt{2}}{9}.$$

(6) If $\cos\alpha=\frac{3}{4}$, $\sin\alpha=\frac{\sqrt{7}}{4}$.

If $\cos\beta=\frac{2}{5}$, $\sin\beta=\frac{\sqrt{21}}{5}$.

$$\therefore\ \sin(\alpha-\beta)=\frac{\sqrt{7}}{4}\cdot\frac{2}{5}-\frac{3}{4}\cdot\frac{\sqrt{21}}{5}=\frac{2\sqrt{7}-3\sqrt{21}}{20}.$$

(7) If $\sin\alpha=\cdot 5=\frac{1}{2}$, $\cos\alpha=\frac{\sqrt{3}}{2}$.

If $\cos\beta=\frac{1}{\sqrt{2}}$, $\sin\beta=\frac{1}{\sqrt{2}}$;

$$\therefore\ \cos(\alpha+\beta)=\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}-\frac{1}{2}\cdot\frac{1}{\sqrt{2}}=\frac{\sqrt{3}-1}{2\sqrt{2}}.$$

(8) If $\cos\alpha=\cdot 0\dot{3}=\frac{1}{30}$, $\sin\alpha=\frac{\sqrt{899}}{30}$.

If $\sin\beta=\frac{1}{2}$, $\cos\beta=\frac{\sqrt{3}}{2}$;

$$\therefore\ \cos(\alpha-\beta)=\frac{1}{30}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{899}}{30}\cdot\frac{1}{2}=\frac{\sqrt{3}+\sqrt{899}}{60}.$$

EXAMPLES—XXIX. (p. 88).

(1) $\cos(90^\circ+A)=\cos 90^\circ.\cos A-\sin 90^\circ.\sin A$
$\qquad =0.\cos A-1.\sin A=-\sin A.$

(2) $\sin(180^\circ+A)=\sin 180^\circ.\cos A+\cos 180^\circ.\sin A$
$\qquad =0.\cos A-1.\sin A=-\sin A.$

(3) $\cos(\pi+\theta)=\cos\pi.\cos\theta-\sin\pi.\sin\theta$
$\qquad =-1.\cos\theta-0.\sin\theta=-\cos\theta.$

(4) $\sin\left(\frac{3\pi}{2}+\theta\right)=\sin\frac{3\pi}{2}\cdot\cos\theta+\cos\frac{3\pi}{2}\cdot\sin\theta$
$\qquad =-1.\cos\theta+0.\sin\theta=-\cos\theta.$

(5) $\operatorname{cosec}\left(\frac{\pi}{2}+a\right)=\frac{1}{\sin\left(\frac{\pi}{2}+a\right)}$

$$=\frac{1}{\sin\frac{\pi}{2}\cdot\cos a+\cos\frac{\pi}{2}\cdot\sin a}$$

$$=\frac{1}{1\,.\,\cos a+0\,.\,\sin a}=\frac{1}{\cos a}=\sec a.$$

(6) $\tan(\pi+a)=\frac{\sin(\pi+a)}{\cos(\pi+a)}=\frac{0\,.\,\cos a-1\,.\,\sin a}{-1\,.\,\cos a-0\,.\,\sin a}=\frac{-\sin a}{-\cos a}=\tan a.$

(7) $\sin(2\pi-\theta)=\sin 2\pi\,.\,\cos\theta-\cos 2\pi\,.\,\sin\theta$

$$=0\,.\,\cos\theta-1\,.\,\sin\theta=-\sin\theta.$$

(8) $\tan(2\pi-\theta)=\frac{\sin(2\pi-\theta)}{\cos(2\pi-\theta)}=\frac{0\,.\,\cos\theta-1\,.\,\sin\theta}{1\,.\,\cos\theta+0\,.\,\sin\theta}=\frac{-\sin\theta}{\cos\theta}=-\tan\theta.$

(9) $\sec(180°-\theta)=\frac{1}{\cos(180°-\theta)}=\frac{1}{-1\,.\,\cos\theta+0\,.\,\sin\theta}=-\frac{1}{\cos\theta}=-\sec\theta.$

(10) $\operatorname{cosec}(\pi-\theta)=\frac{1}{\sin(\pi-\theta)}=\frac{1}{0\,.\,\cos\theta-(-1\,.\,\sin\theta)}=\frac{1}{\sin\theta}=\operatorname{cosec}\theta.$

EXAMPLES—XXX. (p. 89).

(1) $$\sin\theta-\cos\theta=0.$$

$$\sin\theta\cdot\frac{1}{\sqrt{2}}-\cos\theta\cdot\frac{1}{\sqrt{2}}=0$$

$$\sin\theta\,.\,\cos 45°-\cos\theta\,.\,\sin 45°=0;$$

$$\therefore\ \sin(\theta-45°)=0,\ \therefore\ \theta-45°=0°,\text{ or }\theta=45°.$$

(2) $$\sin\theta+\cos\theta=1$$

$$\sin\theta\cdot\frac{1}{\sqrt{2}}+\cos\theta\cdot\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{2}};$$

$$\sin\theta\,.\,\cos 45°+\cos\theta\,.\,\sin 45°=\frac{1}{\sqrt{2}};$$

$$\therefore\ \sin(\theta+45°)=\sin 45°;$$

$$\therefore\ \theta+45°=45°,\text{ or, }\theta=0°.$$

(3) $$\sin\theta - \cos\theta = \sqrt{\frac{3}{2}}$$

$$\sin\theta \cdot \frac{1}{\sqrt{2}} - \cos\theta \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2};$$

$$\therefore\ \sin(\theta - 45^\circ) = \sin 60^\circ;$$

$$\therefore\ \theta - 45^\circ = 60^\circ, \text{ or, } \theta = 105^\circ.$$

(4) $$\sin\theta + \cos\theta = \frac{\sqrt{3}+1}{2}$$

$$\sin\theta \cdot \frac{1}{\sqrt{2}} + \cos\theta \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\sin(\theta + 45^\circ) = \sin 75^\circ, \text{ whence } \theta = 30^\circ, \text{ or,}$$

$$\cos(\theta - 45^\circ) = \cos 15^\circ, \text{ whence } \theta = 60^\circ, \text{ or,}$$

$$\cos(45^\circ - \theta) = \cos 15^\circ, \text{ whence } \theta = -30^\circ.$$

(5) $$\sin\theta + \cos\theta = \sqrt{2}$$

$$\sin\theta \cdot \frac{1}{\sqrt{2}} + \cos\theta \cdot \frac{1}{\sqrt{2}} = 1$$

$$\sin(\theta + 45^\circ) = \sin 90^\circ, \text{ or, } \theta = 45^\circ.$$

(6) $$\sin\theta - \cos\theta = \frac{\sqrt{3}-1}{2}$$

$$\sin\theta \cdot \frac{1}{\sqrt{2}} - \cos\theta \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\sin(\theta - 45^\circ) = \sin 15^\circ, \text{ whence } \theta = 60^\circ.$$

EXAMPLES—XXXI. (p. 92).

(1) $\sin 6A + \sin 4A = 2\sin\frac{6A+4A}{2} \cdot \cos\frac{6A-4A}{2} = 2\sin 5A \,.\, \cos A.$

(2) $\sin 5A - \sin 3A = 2\cos\frac{5A+3A}{2} \cdot \sin\frac{5A-3A}{2} = 2\cos 4A \,.\, \sin A.$

(3) $\cos 7\theta + \cos 9\theta = 2\cos\frac{7\theta+9\theta}{2} \cdot \cos\frac{9\theta-7\theta}{2} = 2\cos 8\theta \,.\, \cos\theta.$

(4) $\cos\theta - \cos 5\theta = 2\sin\frac{\theta+5\theta}{2} \cdot \sin\frac{5\theta-\theta}{2} = 2\sin 3\theta \,.\, \sin 2\theta.$

(5) $\sin a + \sin 4a = 2\sin\frac{a+4a}{2}\cdot\cos\frac{4a-a}{2} = 2\sin\frac{5a}{2}\cdot\cos\frac{3a}{2}.$

(6) $\cos 5a - \cos 8a = 2\sin\frac{5a+8a}{2}\cdot\sin\frac{8a-5a}{2} = 2\sin\frac{13a}{2}\cdot\sin\frac{3a}{2}.$

(7) $2\sin 5\theta\,.\,\cos 7\theta = \sin(5\theta+7\theta) - \sin(7\theta-5\theta) = \sin 12\theta - \sin 2\theta.$

(8) $2\sin 3\theta\,.\,\sin 5\theta = \cos(5\theta-3\theta) - \cos(5\theta+3\theta) = \cos 2\theta - \cos 8\theta.$

(9) $2\cos a\,.\,\cos 4a = \cos(a+4a) + \cos(4a-a) = \cos 5a + \cos 3a.$

(10) $2\cos a\,.\,\sin 2a = \sin(a+2a) + \sin(2a-a) = \sin 3a + \sin a.$

(11) $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \dfrac{2\sin\frac{A+B}{2}\cdot\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}\cdot\cos\frac{A-B}{2}} = \dfrac{\sin\frac{A+B}{2}}{\cos\frac{A+B}{2}} = \tan\dfrac{A+B}{2}.$

(12) $\dfrac{\cos A - \cos 3A}{\sin 3A - \sin A} = \dfrac{2\sin 2A\,.\,\sin A}{2\cos 2A\,.\,\sin A} = \dfrac{\sin 2A}{\cos 2A} = \tan 2A.$

(13) $\dfrac{\sin 2A + \sin A}{\cos 2A + \cos A} = \dfrac{2\sin\frac{3A}{2}\cdot\cos\frac{A}{2}}{2\cos\frac{3A}{2}\cdot\cos\frac{A}{2}} = \dfrac{\sin\frac{3A}{2}}{\cos\frac{3A}{2}} = \tan\dfrac{3A}{2}.$

(14) $\cos(30^\circ-\theta) - \cos(30^\circ+\theta) = 2\sin 30^\circ.\,\sin\theta = 2\times\frac{1}{2}\cdot\sin\theta = \sin\theta.$

(15) $\cos\left(\frac{\pi}{3}+\theta\right) + \cos\left(\frac{\pi}{3}-\theta\right) = 2\cos\frac{\pi}{3}\cdot\cos\theta = 2\times\frac{1}{2}\cdot\cos\theta = \cos\theta.$

(16) $\sin\left(\frac{\pi}{3}+a\right) - \sin\left(\frac{\pi}{3}-a\right) = 2\cos\frac{\pi}{3}\cdot\sin a = 2\times\frac{1}{2}\cdot\sin a = \sin a.$

(17) $\dfrac{\sin a - \sin\beta}{\cos\beta - \cos a} = \dfrac{2\cos\frac{a+\beta}{2}\cdot\sin\frac{a-\beta}{2}}{2\sin\frac{a+\beta}{2}\cdot\sin\frac{a-\beta}{2}} = \dfrac{\cos\frac{a+\beta}{2}}{\sin\frac{a+\beta}{2}} = \cot\dfrac{a+\beta}{2}.$

(18) $$\frac{\sin\alpha-\sin\beta}{\cos\beta+\cos\alpha}=\frac{2\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}}{2\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}}=\frac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha-\beta}{2}}=\tan\frac{\alpha-\beta}{2}.$$

(19) $$\frac{\sin5\theta+\sin3\theta}{\cos3\theta-\cos5\theta}=\frac{2\sin4\theta.\cos\theta}{2\sin4\theta.\sin\theta}=\frac{\cos\theta}{\sin\theta}=\cot\theta.$$

(20) $$\frac{\cos\alpha+\cos\beta}{\cos\beta-\cos\alpha}=\frac{2\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}}{2\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}}=\frac{\cos\frac{\alpha+\beta}{2}}{\sin\frac{\alpha+\beta}{2}}\div\frac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha-\beta}{2}}=\frac{\cot\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}}.$$

EXAMPLES—XXXII. (p. 93).

(1)
$$\sin\alpha-\cos\beta=\sin\alpha-\sin\left(\frac{\pi}{2}-\beta\right)=2\cos\frac{1}{2}\left(\alpha+\frac{\pi}{2}-\beta\right).\sin\frac{1}{2}\left(\alpha-\frac{\pi}{2}+\beta\right).$$

(2)
$$\sin\left(\frac{\pi}{2}+\alpha\right)+\cos\left(\frac{\pi}{2}-\alpha\right)=\sin\left(\frac{\pi}{2}+\alpha\right)+\sin\alpha=2\sin\left(\frac{\pi}{4}+\alpha\right)\cdot\cos\frac{\pi}{4}.$$

(3) $\sin\alpha+\cos\alpha=\sin\alpha+\sin\left(\frac{\pi}{2}-\alpha\right)=2\sin\frac{\pi}{4}\cdot\cos\left(\alpha-\frac{\pi}{4}\right).$

(4) $\sin\alpha-\cos\alpha=\sin\alpha-\sin\left(\frac{\pi}{2}-\alpha\right)=2\cos\frac{\pi}{4}\cdot\sin\left(\alpha-\frac{\pi}{4}\right).$

(5) $\sin30^\circ+\cos80^\circ=\sin30^\circ+\sin10^\circ=2\sin20^\circ.\cos10^\circ.$

(6) $\sin20^\circ-\cos80^\circ=\sin20^\circ-\sin10^\circ=2\cos15^\circ.\sin5^\circ.$

(7) $\sin\frac{\pi}{4}+\cos\frac{\pi}{6}=\sin\frac{\pi}{4}+\sin\frac{\pi}{3}=2\sin\frac{7\pi}{24}\cdot\cos\frac{\pi}{24}.$

(8) $\sin\frac{\pi}{3}-\cos\frac{\pi}{5}=\sin\frac{\pi}{3}-\sin\frac{3\pi}{10}=2\cos\frac{19\pi}{60}\cdot\sin\frac{\pi}{60}.$

EXAMPLES—XXXIII. (p. 96).

(1) $$\frac{\tan\alpha+\tan\beta}{\cot\alpha+\cot\beta}=\frac{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}{\dfrac{\cos\alpha}{\sin\alpha}+\dfrac{\cos\beta}{\sin\beta}}=\frac{\dfrac{\sin\alpha\,.\cos\beta+\cos\alpha\,.\sin\beta}{\cos\alpha\,.\cos\beta}}{\dfrac{\cos\alpha\,.\sin\beta+\sin\alpha\,.\cos\beta}{\sin\alpha\,.\sin\beta}}.$$

$$=\frac{\sin\alpha\,.\sin\beta}{\cos\alpha\,.\cos\beta}=\tan\alpha\,.\tan\beta.$$

(2) $$\frac{\tan\alpha+\tan\beta}{\cot\alpha-\tan\beta}=\frac{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}{\dfrac{\cos\alpha}{\sin\alpha}-\dfrac{\sin\beta}{\cos\beta}}=\frac{\dfrac{\sin\alpha\,.\cos\beta+\cos\alpha\,.\sin\beta}{\cos\alpha\,.\cos\beta}}{\dfrac{\cos\alpha\,.\cos\beta-\sin\alpha\,.\sin\beta}{\sin\alpha\,.\cos\beta}}.$$

$$=\frac{\sin(\alpha+\beta)\,.\sin\alpha\,.\cos\beta}{\cos(\alpha+\beta)\,.\cos\alpha\,.\cos\beta}=\tan(\alpha+\beta)\,.\tan\alpha.$$

(3) $$\frac{\tan\alpha-\tan\beta}{\cot\alpha+\tan\beta}=\frac{\dfrac{\sin\alpha}{\cos\alpha}-\dfrac{\sin\beta}{\cos\beta}}{\dfrac{\cos\alpha}{\sin\alpha}+\dfrac{\sin\beta}{\cos\beta}}=\frac{\dfrac{\sin\alpha\,.\cos\beta-\cos\alpha\,.\sin\beta}{\cos\alpha\,.\cos\beta}}{\dfrac{\cos\alpha\,.\cos\beta+\sin\alpha\,.\sin\beta}{\sin\alpha\,.\cos\beta}}.$$

$$=\frac{\sin(\alpha-\beta)\,.\sin\alpha\,.\cos\beta}{\cos(\alpha-\beta)\,.\cos\alpha\,.\cos\beta}=\tan(\alpha-\beta)\,.\tan\alpha.$$

(4) $$\tan\frac{\phi+\psi}{2}+\tan\frac{\phi-\psi}{2}=\frac{\sin\dfrac{\phi+\psi}{2}}{\cos\dfrac{\phi+\psi}{2}}+\frac{\sin\dfrac{\phi-\psi}{2}}{\cos\dfrac{\phi-\psi}{2}}$$

$$=\frac{\sin\dfrac{\phi+\psi}{2}\cdot\cos\dfrac{\phi-\psi}{2}+\cos\dfrac{\phi+\psi}{2}\cdot\sin\dfrac{\phi-\psi}{2}}{\cos\dfrac{\phi+\psi}{2}\cdot\cos\dfrac{\phi-\psi}{2}}$$

$$=\frac{\sin\left(\dfrac{\phi+\psi}{2}+\dfrac{\phi-\psi}{2}\right)}{\dfrac{1}{2}(\cos\phi+\cos\psi)}=\frac{\sin\phi}{\dfrac{1}{2}(\cos\phi+\cos\psi)}=\frac{2\sin\phi}{\cos\phi+\cos\psi}.$$

(5) $\sin\phi = \sin\{\psi + (\phi - \psi)\} = \sin\psi \,.\, \cos(\phi - \psi) + \cos\psi \,.\, \sin(\phi - \psi)$.

(6) $\cos\phi = \cos\{(\phi + \psi) - \psi\} = \cos(\phi + \psi) \,.\, \cos\psi + \sin(\phi + \psi) \,.\, \sin\psi$.

(7) $(\cos\alpha + \cos\beta)\{1 - \cos(\alpha + \beta)\} = (\cos\alpha + \cos\beta)(1 - \cos\alpha.\cos\beta + \sin\alpha.\sin\beta)$

$= \cos\alpha + \cos\beta - \cos^2\alpha.\cos\beta - \cos\alpha.\cos^2\beta + \sin\alpha.\sin\beta.\cos\alpha + \sin\alpha\,\sin\beta.\cos\beta$

$= \cos\alpha(1 - \cos^2\beta) + \cos\beta(1 - \cos^2\alpha) + \sin\alpha.\sin\beta.\cos\alpha + \sin\alpha.\sin\beta.\cos\beta$

$= \cos\alpha.\sin^2\beta + \cos\beta.\sin^2\alpha + \sin\alpha.\sin\beta.\cos\alpha + \sin\alpha.\sin\beta.\cos\beta$

$= \sin\beta(\cos\alpha.\sin\beta) + \sin\alpha(\cos\beta.\sin\alpha) + \sin\alpha(\sin\beta.\cos\alpha) + \sin\beta.(\sin\alpha.\cos\beta)$

$= \sin\beta.(\cos\alpha.\sin\beta + \sin\alpha.\cos\beta) + \sin\alpha(\cos\beta.\sin\alpha. + \sin\beta.\cos\alpha)$

$= \sin\beta.\sin(\alpha + \beta) + \sin\alpha.\sin(\alpha + \beta)$

$= (\sin\alpha + \sin\beta).\sin(\alpha + \beta)$.

(8)

$$\frac{\sin(\alpha+\beta)}{\sin\alpha+\sin\beta} = \frac{\sin\left(\frac{\alpha+\beta}{2} + \frac{\alpha+\beta}{2}\right)}{\sin\alpha+\sin\beta} = \frac{\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha+\beta}{2} + \cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha+\beta}{2}}{2\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}}$$

$$= \frac{2\cos\frac{\alpha+\beta}{2}}{2\cos\frac{\alpha-\beta}{2}} = \frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}.$$

(9)

$$\frac{\sin(\alpha+\beta)}{\sin\alpha-\sin\beta} = \frac{\sin\left(\frac{\alpha+\beta}{2} + \frac{\alpha+\beta}{2}\right)}{\sin\alpha-\sin\beta} = \frac{\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha+\beta}{2} + \cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha+\beta}{2}}{2\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}}$$

$$= \frac{2\sin\frac{\alpha+\beta}{2}}{2\sin\frac{\alpha-\beta}{2}} = \frac{\sin\frac{\alpha+\beta}{2}}{\sin\frac{\alpha-\beta}{2}}.$$

(10) $\cot\frac{\alpha+\beta}{2}+\cot\frac{\alpha-\beta}{2}=\frac{\cos\frac{\alpha+\beta}{2}}{\sin\frac{\alpha+\beta}{2}}+\frac{\cos\frac{\alpha-\beta}{2}}{\sin\frac{\alpha-\beta}{2}}$

$$=\frac{\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}+\cos\frac{\alpha-\beta}{2}\cdot\sin\frac{\alpha+\beta}{2}}{\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}}$$

$$=\frac{\sin\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right)}{\frac{1}{2}(\cos\beta-\cos\alpha)}=\frac{\sin\alpha}{\frac{1}{2}(\cos\beta-\cos\alpha)}=\frac{2\sin\alpha}{\cos\beta-\cos\alpha}.$$

(11) $\tan\frac{\alpha+\beta}{2}-\tan\frac{\alpha-\beta}{2}=\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha+\beta}{2}}-\frac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$

$$=\frac{\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}-\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}}$$

$$=\frac{\sin\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right)}{\frac{1}{2}(\cos\alpha+\cos\beta)}=\frac{\sin\beta}{\frac{1}{2}(\cos\alpha+\cos\beta)}=\frac{2\sin\beta}{\cos\alpha+\cos\beta}.$$

(12) $\frac{\cos\alpha-\cos\beta}{\sin\alpha+\sin\beta}=\frac{2\sin\frac{\beta+\alpha}{2}\cdot\sin\frac{\beta-\alpha}{2}}{2\sin\frac{\beta+\alpha}{2}\cdot\cos\frac{\beta-\alpha}{2}}=\tan\frac{\beta-\alpha}{2}.$

(13) $\cot\beta-\tan\alpha=\frac{\cos\beta}{\sin\beta}-\frac{\sin\alpha}{\cos\alpha}=\frac{\cos\alpha.\cos\beta-\sin\alpha.\sin\beta}{\cos\alpha.\sin\beta}=\frac{\cos(\alpha+\beta)}{\cos\alpha.\sin\beta}.$

(14) $\cot\theta+\tan\phi=\frac{\cos\theta}{\sin\theta}+\frac{\sin\phi}{\cos\phi}=\frac{\cos\theta.\cos\phi+\sin\theta.\sin\phi}{\sin\theta.\cos\phi}=\frac{\cos(\phi-\theta)}{\sin\theta.\cos\phi}.$

(15) $\tan^2\alpha - \tan^2\beta = \dfrac{\sin^2\alpha}{\cos^2\alpha} - \dfrac{\sin^2\beta}{\cos^2\beta} = \dfrac{\sin^2\alpha \, . \, \cos^2\beta - \cos^2\alpha \, . \, \sin^2\beta}{\cos^2\alpha \, . \, \cos^2\beta}$

$$= \frac{(\sin\alpha \, . \, \cos\beta + \cos\alpha \, . \, \sin\beta)(\sin\alpha \, . \, \cos\beta - \cos\alpha \, . \, \sin\beta)}{\cos^2\alpha \, . \, \cos^2\beta}$$

$$= \frac{\sin(\alpha+\beta) \, . \, \sin(\alpha-\beta)}{\cos^2\alpha \, . \, \cos^2\beta}.$$

(16) $1 + \tan\alpha.\tan\beta = 1 + \dfrac{\sin\alpha.\sin\beta}{\cos\alpha.\cos\beta} = \dfrac{\cos\alpha.\cos\beta + \sin\alpha.\sin\beta}{\cos\alpha \, . \, \cos\beta} = \dfrac{\cos(\alpha-\beta)}{\cos\alpha.\cos\beta}.$

(17) $1 - \tan\alpha \, . \, \tan\beta = 1 - \dfrac{\sin\alpha \, . \, \sin\beta}{\cos\alpha \, . \, \cos\beta} = \dfrac{\cos\alpha \, . \, \cos\beta - \sin\alpha \, . \, \sin\beta}{\cos\alpha \, . \, \cos\beta}$

$$= \frac{\cos(\alpha+\beta)}{\cos\alpha \, . \, \cos\beta}.$$

(18) $\dfrac{\cot\alpha + \tan\beta}{\tan\alpha + \cot\beta} = \dfrac{\dfrac{\cos\alpha}{\sin\alpha} + \dfrac{\sin\beta}{\cos\beta}}{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\cos\beta}{\sin\beta}} = \dfrac{\dfrac{\cos\alpha \, . \, \cos\beta + \sin\alpha \, . \, \sin\beta}{\sin\alpha \, . \, \cos\beta}}{\dfrac{\sin\alpha \, . \, \sin\beta + \cos\alpha \, . \, \cos\beta}{\cos\alpha \, . \, \sin\beta}}$

$$= \frac{\cos\alpha \, . \, \sin\beta}{\sin\alpha \, . \, \cos\beta} = \cot\alpha \, . \, \tan\beta.$$

(19) $\dfrac{\tan^2 x - \tan^2 y}{1 - \tan^2 x \, . \, \tan^2 y} = \dfrac{\dfrac{\sin^2 x}{\cos^2 x} - \dfrac{\sin^2 y}{\cos^2 y}}{1 - \dfrac{\sin^2 x \, . \, \sin^2 y}{\cos^2 x \, . \, \cos^2 y}} = \dfrac{\sin^2 x \, . \, \cos^2 y - \cos^2 x \, . \, \sin^2 y}{\cos^2 x \, . \, \cos^2 y - \sin^2 x \, . \, \sin^2 y}$

$$= \frac{(\sin x \, . \, \cos y + \cos x \, . \, \sin y)(\sin x \, . \, \cos y - \cos x \, . \, \sin y)}{(\cos x \, . \, \cos y + \sin x \, . \, \sin y)(\cos x \, . \, \cos y - \sin x \, . \, \sin y)}$$

$$= \frac{\sin(x+y) \, . \, \sin(x-y)}{\cos(x-y) \, . \, \cos(x+y)} = \tan(x+y) \, . \, \tan(x-y).$$

(20) $\cot(\theta + 45°) = \dfrac{\cos(\theta+45°)}{\sin(\theta+45°)} = \dfrac{\cos\theta \cdot \dfrac{1}{\sqrt{2}} - \sin\theta \cdot \dfrac{1}{\sqrt{2}}}{\sin\theta \cdot \dfrac{1}{\sqrt{2}} + \cos\theta \cdot \dfrac{1}{\sqrt{2}}} = \dfrac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta}$

$$= \frac{\dfrac{\cos\theta}{\sin\theta} - 1}{1 + \dfrac{\cos\theta}{\sin\theta}} = \frac{\cot\theta - 1}{\cot\theta + 1}.$$

(21) $\sin\theta-\cos\theta=\sqrt{2}\left(\sin\theta\cdot\frac{1}{\sqrt{2}}-\cos\theta\cdot\frac{1}{\sqrt{2}}\right)=\sqrt{2}\,.\sin(45^\circ-\theta).$

(22) $\cos\theta-\sin\theta=\sqrt{2}\left(\cos\theta\cdot\frac{1}{\sqrt{2}}-\sin\theta\cdot\frac{1}{\sqrt{2}}\right)=\sqrt{2}\sin\left(\frac{\pi}{4}-\theta\right).$

(23) $$\frac{\tan\alpha-\tan\beta}{\tan\alpha+\tan\beta}=\frac{\dfrac{\sin\alpha}{\cos\alpha}-\dfrac{\sin\beta}{\cos\beta}}{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}=\frac{\sin\alpha\,.\cos\beta-\cos\alpha\,.\sin\beta}{\sin\alpha\,.\cos\beta+\cos\alpha\,.\sin\beta}=\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}.$$

(24) $$\frac{\cot x-\cot y}{\cot x+\cot y}=\frac{\dfrac{\cos x}{\sin x}-\dfrac{\cos y}{\sin y}}{\dfrac{\cos x}{\sin x}+\dfrac{\cos y}{\sin y}}=\frac{\cos x\,.\sin y-\sin x\,.\cos y}{\cos x\,.\sin y+\sin x\,.\cos y}=\frac{\sin(y-x)}{\sin(y+x)}.$$

(25) $$\begin{aligned}\cos(A-B)+\sin(A+B)&=\cos(A-B)+\cos(90^\circ-A-B)\\&=2\cos(45^\circ-B)\,.\cos(45^\circ-A)\\&=2\cos(B-45^\circ)\,.\sin(45^\circ+A).\end{aligned}$$

(26) $$\begin{aligned}\cos(A-B)-\sin(A+B)&=\sin(90^\circ-A+B)-\sin(A+B)\\&=2\cos(45^\circ+B)\,.\sin(45^\circ-A).\end{aligned}$$

(27) $$\begin{aligned}\cos(A+B)+\sin(A-B)&=\cos(A+B)+\cos(90^\circ-A+B)\\&=2\cos(45^\circ+B)\,.\cos(45^\circ-A)\\&=2\cos(45^\circ+B)\,.\sin(45^\circ+A).\end{aligned}$$

(28) $$\begin{aligned}\cos(A+B)-\sin(A-B)&=\sin(90^\circ-A-B)-\sin(A-B)\\&=2\cos(45^\circ-B)\,.\sin(45^\circ-A).\end{aligned}$$

(29) $$\begin{aligned}\frac{\cos\alpha+\cos\beta}{\cos\alpha-\cos\beta}&=\frac{\cos\alpha+\cos\beta}{-(\cos\beta-\cos\alpha)}\\&=-\frac{2\cos\dfrac{\alpha+\beta}{2}\,.\cos\dfrac{\alpha-\beta}{2}}{2\sin\dfrac{\alpha+\beta}{2}\,.\sin\dfrac{\alpha-\beta}{2}}=-\frac{\cot\dfrac{\alpha+\beta}{2}}{\tan\dfrac{\alpha-\beta}{2}}.\end{aligned}$$

(30) $$\begin{aligned}\sec72^\circ-\sec36^\circ&=\frac{1}{\cos72^\circ}-\frac{1}{\cos36^\circ}=\frac{\cos36^\circ-\cos72^\circ}{\cos72^\circ.\cos36^\circ}\\&=\frac{2\sin54^\circ.\sin18^\circ}{\sin18^\circ.\sin54^\circ}=2=\sec60^\circ.\end{aligned}$$

(31) $(\sin 81^\circ+\sin 9^\circ)(\sin 81^\circ-\sin 9^\circ)$

$$=(2\sin 45^\circ.\cos 36^\circ).(2\cos 45^\circ.\sin 36^\circ)$$
$$=2\cdot\frac{1}{\sqrt{2}}\cdot\sin 54^\circ.\,2\cdot\frac{1}{\sqrt{2}}\cdot\cos 54^\circ$$
$$=2\sin 54^\circ.\cos 54^\circ$$
$$=\sin 108^\circ.$$

(32) $\dfrac{\cos 3^\circ-\cos 33^\circ}{\sin 3^\circ+\sin 33^\circ}=\dfrac{2\sin 18^\circ.\sin 15^\circ}{2\sin 18^\circ.\cos 15^\circ}=\tan 15^\circ.$

(33) $\dfrac{\sin 33^\circ+\sin 3^\circ}{\cos 33^\circ+\cos 3^\circ}=\dfrac{2\sin 18^\circ.\cos 15^\circ}{2\cos 18^\circ.\cos 15^\circ}=\tan 18^\circ.$

(34) $\dfrac{\cos 9^\circ+\sin 9^\circ}{\cos 9^\circ-\sin 9^\circ}=\dfrac{\sin 81^\circ+\sin 9^\circ}{\sin 81^\circ-\sin 9^\circ}=\dfrac{2\sin 45^\circ.\cos 36^\circ}{2\cos 45^\circ.\sin 36^\circ}=\cot 36^\circ=\tan 54^\circ.$

(35) $\dfrac{\cos 27^\circ-\sin 27^\circ}{\cos 27^\circ+\sin 27^\circ}=\dfrac{\sin 63^\circ-\sin 27^\circ}{\sin 63^\circ+\sin 27^\circ}=\dfrac{2\cos 45^\circ.\sin 18^\circ}{2\sin 45^\circ.\cos 18^\circ}=\tan 18^\circ.$

(36) $\tan 50^\circ+\cot 50^\circ=\tan 50^\circ+\tan 40^\circ$

$$=\frac{\sin 50^\circ.\cos 40^\circ+\cos 50^\circ.\sin 40^\circ}{\cos 50^\circ.\cos 40^\circ}=\frac{\sin 90^\circ}{\frac{1}{2}\{\cos 90^\circ+\cos 10^\circ\}}$$
$$=\frac{2\sin 90^\circ}{\cos 10^\circ}=\frac{2}{\cos 10^\circ}=2\sec 10^\circ.$$

EXAMPLES—XXXIV. (p. 100).

(1) $\dfrac{2\cot A}{1+\cot^2 A}=\dfrac{2\cot A}{\operatorname{cosec}^2 A}=\dfrac{2\cos A}{\sin A}\cdot\sin^2 A=2\cos A.\sin A=\sin 2A.$

(2) $\dfrac{\sin 2A}{1+\cos 2A}\cdot\dfrac{\cos A}{1+\cos A}=\dfrac{2\sin A.\cos A}{2\cos^2 A}\cdot\dfrac{\cos A}{1+\cos A}=\dfrac{\sin A}{1+\cos A}$

$$=\frac{2\sin\frac{A}{2}\cdot\cos\frac{A}{2}}{2\cos^2\frac{A}{2}}=\tan\frac{A}{2}.$$

(3) $\operatorname{cosec} A+\cot A=\dfrac{1}{\sin A}+\dfrac{\cos A}{\sin A}=\dfrac{1+\cos A}{\sin A}=\dfrac{2\cos^2\frac{A}{2}}{2\sin\frac{A}{2}\cdot\cos\frac{A}{2}}=\cot\dfrac{A}{2}.$

(4)

$$\tan\theta+\cot\theta=\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}=\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\,.\,\cos\theta}=\frac{1}{\sin\theta\,.\cos\theta}=\frac{2}{2\sin\theta.\cos\theta}=\frac{2}{\sin 2\theta}.$$

(5) $$\frac{2\tan\theta}{1+\tan^2\theta}=\frac{2\dfrac{\sin\theta}{\cos\theta}}{\sec^2\theta}=\frac{2\sin\theta}{\cos\theta}\cdot\cos^2\theta=2\sin\theta\,.\,\cos\theta=\sin 2\theta.$$

(6) $$2\operatorname{cosec}2A=\frac{2}{\sin 2A}=\frac{2}{2\sin A.\cos A}=\operatorname{cosec}A\,.\,\sec A.$$

(7) $$\frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{1-\dfrac{\sin^2\theta}{\cos^2\theta}}{1+\dfrac{\sin^2\theta}{\cos^2\theta}}=\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}=\cos^2\theta-\sin^2\theta=\cos 2\theta.$$

(8) $$\frac{2\sec 2\theta}{1+\sec 2\theta}=\frac{\dfrac{2}{\cos 2\theta}}{1+\dfrac{1}{\cos 2\theta}}=\frac{2}{\cos 2\theta+1}=\frac{2}{2\cos^2\theta}=\sec^2\theta.$$

(9) $$\frac{1-\tan A}{1+\tan A}=\frac{\cos A-\sin A}{\cos A+\sin A}=\frac{\cos^2A-\sin^2A}{(\cos A+\sin A)^2}=\frac{1-2\sin^2A}{1+\sin 2A}.$$

(10) $$\cot\theta-2\cot 2\theta=\frac{\cos\theta}{\sin\theta}-\frac{2\cos 2\theta}{\sin 2\theta}=\frac{\cos\theta}{\sin\theta}-\frac{\cos 2\theta}{\sin\theta\,.\,\cos\theta}.$$

$$=\frac{\cos^2\theta-\cos 2\theta}{\sin\theta\,.\,\cos\theta}=\frac{\cos^2\theta-2\cos^2\theta+1}{\sin\theta\ \cos\theta}=\frac{\sin^2\theta}{\sin\theta\,.\,\cos\theta}=\tan\theta.$$

(11) $$\frac{1-\cos a}{\sin a}=\frac{2\sin^2\dfrac{a}{2}}{2\sin\dfrac{a}{2}\cdot\cos\dfrac{a}{2}}=\frac{\sin\dfrac{a}{2}}{\cos\dfrac{a}{2}}=\tan\frac{a}{2}\cdot$$

(12) $$\frac{2\sqrt{(\operatorname{cosec}^2\phi-1)}}{\operatorname{cosec}^2\phi}=\frac{2\,.\,\cot\phi}{\operatorname{cosec}^2\phi}=\frac{2.\cos\phi.\sin^2\phi}{\sin\phi}=2\sin\phi\,.\,\cos\phi=\sin 2\phi.$$

(13) $$\frac{2-\sec^2\phi}{\sec^2\phi}=2\cos^2\phi-1=\cos 2\phi.$$

(14) $\frac{2\cot\phi}{\cot^2\phi-1}=\frac{2\cos\phi\,.\,\sin\phi}{\cos^2\phi-\sin^2\phi}=\frac{\sin2\phi}{\cos2\phi}=\tan2\phi.$

(15) $\sqrt{\left(\frac{\sec2a-1}{2\sec2a}\right)}=\sqrt{\left(\frac{1-\cos2a}{2}\right)}=\sqrt{\left(\frac{1-1+2\sin^2a}{2}\right)}=\sin a.$

(16) $\sqrt{\left(\frac{\sec2a+1}{2\sec2a}\right)}=\sqrt{\left(\frac{1+\cos2a}{2}\right)}=\sqrt{\left(\frac{1+2\cos^2a-1}{2}\right)}=\cos a.$

(17) $\operatorname{cosec}2a-\cot2a=\frac{1-\cos2a}{\sin2a}=\frac{2\sin^2a}{2\sin a\,.\,\cos a}=\tan a.$

(18) $\operatorname{cosec}2\beta+\cot2\beta=\frac{1+\cos2\beta}{\sin2\beta}=\frac{2\cos^2\beta}{2\sin\beta\,.\,\cos\beta}=\cot\beta.$

(19) $\tan(45^\circ+A)=\frac{\tan45^\circ+\tan A}{1-\tan45^\circ.\,\tan A}=\frac{1+\tan A}{1-\tan A}=\frac{\cos A+\sin A}{\cos A-\sin A}$

$$=\frac{\cos^2A-\sin^2A}{(\cos A-\sin A)^2}=\frac{\cos2A}{1-\sin2A}.$$

(20) $\cot(45^\circ-A)=\frac{1}{\tan(45^\circ-A)}=\frac{1}{\frac{\tan45^\circ-\tan A}{1+\tan45^\circ.\,\tan A}}=\frac{1+\tan A}{1-\tan A}$

$$=\frac{\cos A+\sin A}{\cos A-\sin A}=\frac{(\cos A+\sin A)^2}{\cos^2A-\sin^2A}=\frac{1+\sin2A}{\cos2A}=\sec2A+\tan2A.$$

(21) $\frac{1+\sin a}{1+\cos a}=\frac{\cos^2\frac{a}{2}+\sin^2\frac{a}{2}+2\sin\frac{a}{2}\cdot\cos\frac{a}{2}}{2\cos^2\frac{a}{2}}=\frac{1}{2}+\frac{1}{2}\tan^2\frac{a}{2}+\tan\frac{a}{2}$

$$=\frac{1}{2}\left(1+\tan^2\frac{a}{2}+2\tan\frac{a}{2}\right)=\frac{1}{2}\left(1+\tan\frac{a}{2}\right)^2.$$

(22) $\frac{1-\sin a}{1-\cos a}=\frac{\cos^2\frac{a}{2}+\sin^2\frac{a}{2}-2\sin\frac{a}{2}\cdot\cos\frac{a}{2}}{2\sin^2\frac{a}{2}}=\frac{1}{2}\cot^2\frac{a}{2}+\frac{1}{2}-\cot\frac{a}{2}$

$$=\frac{1}{2}\left(\cot^2\frac{a}{2}+1-2\cot\frac{a}{2}\right)=\frac{1}{2}\left(\cot\frac{a}{2}-1\right)^2.$$

(23) $$\tan\frac{\theta}{2}+\frac{1}{2}\tan\theta\,.\,\sec^2\frac{\theta}{2}=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}+\frac{\sin\theta}{2\cos\theta\,.\,\cos^2\frac{\theta}{2}}$$

$$=\frac{2\cos\theta\,.\,\cos\frac{\theta}{2}\cdot\sin\frac{\theta}{2}+\sin\theta}{2\cos\theta\,.\,\cos^2\frac{\theta}{2}}=\frac{\cos\theta\,.\,\sin\theta+\sin\theta}{2\cos\theta\,.\,\cos^2\frac{\theta}{2}}$$

$$=\frac{\sin\theta(\cos\theta+1)}{2\cos\theta\,.\,\cos^2\frac{\theta}{2}}=\frac{\sin\theta\,.\,2\cos^2\frac{\theta}{2}}{2\cos\theta\,.\,\cos^2\frac{\theta}{2}}=\tan\theta.$$

(24) $$\frac{1+\sin\theta}{1-\sin\theta}=\frac{(1+\sin\theta)^2}{1-\sin^2\theta}=\left(\frac{1+\sin\theta}{\cos\theta}\right)^2=(\sec\theta+\tan\theta)^2.$$

(25) $$\frac{1-\tan^2(45^\circ-A)}{1+\tan^2(45^\circ-A)}=\frac{\cos^2(45^\circ-A)-\sin^2(45^\circ-A)}{\cos^2(45^\circ-A)+\sin^2(45^\circ-A)}$$

$$=\frac{\cos 2(45^\circ-A)}{1}=\cos(90^\circ-2A)=\sin 2A.$$

(26) $$\frac{\tan\left(\frac{\pi}{4}+\theta\right)-\tan\left(\frac{\pi}{4}-\theta\right)}{\tan\left(\frac{\pi}{4}+\theta\right)+\tan\left(\frac{\pi}{4}-\theta\right)}=\frac{\frac{1+\tan\theta}{1-\tan\theta}-\frac{1-\tan\theta}{1+\tan\theta}}{\frac{1+\tan\theta}{1-\tan\theta}+\frac{1-\tan\theta}{1+\tan\theta}}$$

$$=\frac{1+2\tan\theta+\tan^2\theta-1+2\tan\theta-\tan^2\theta}{1+2\tan\theta+\tan^2\theta+1-2\tan\theta+\tan^2\theta}$$

$$=\frac{4\tan\theta}{2+2\tan^2\theta}=\frac{2\tan\theta}{1+\tan^2\theta}=2\sin\theta\,.\,\cos\theta=\sin 2\theta.$$

EXAMPLES—XXXV. (p. 103).

1. (1) $$\frac{\cos 3\theta-\sin 3\theta}{\sin\theta+\cos\theta}=\frac{4\cos^3\theta-3\cos\theta-3\sin\theta+4\sin^3\theta}{\sin\theta+\cos\theta}$$

$$=\frac{4(\sin^3\theta+\cos^3\theta)-3(\sin\theta+\cos\theta)}{\sin\theta+\cos\theta}$$

$$=4(\sin^2\theta-\sin\theta\,.\,\cos\theta+\cos^2\theta)-3$$

$$=1-4\sin\theta\,.\,\cos\theta=1-2\sin 2\theta.$$

(2) $\dfrac{2\tan\theta+\sec\theta}{1+\tan^2\theta}=\dfrac{2\tan\theta+\sec\theta}{\sec^2\theta}=2\tan\theta.\cos^2\theta+\cos\theta=\sin 2\theta+\cos\theta.$

(3) $\tan\dfrac{A}{2}+2\sin^2\dfrac{A}{2}\cot A=\sin\dfrac{A}{2}\left\{\dfrac{1}{\cos\dfrac{A}{2}}+2\sin\dfrac{A}{2}\cdot\dfrac{\cos A}{\sin A}\right\}$

$$=\sin\frac{A}{2}\left\{\frac{1}{\cos\frac{A}{2}}+\frac{\cos A}{\cos\frac{A}{2}}\right\}=\sin\frac{A}{2}\left(\frac{2\cos^2\frac{A}{2}}{\cos\frac{A}{2}}\right)=2\sin\frac{A}{2}\cdot\cos\frac{A}{2}=\sin A.$$

(4) $\dfrac{\cot A}{\cot A-\cot 3A}+\dfrac{\tan A}{\tan A-\tan 3A}$

$$=\frac{\dfrac{1}{\tan A}}{\dfrac{1}{\tan A}-\dfrac{1-3\tan^2A}{\tan A(3-\tan^2A)}}+\frac{\tan A}{\tan A-\dfrac{\tan A(3-\tan^2A)}{1-3\tan^2A}}$$

$$=\frac{1}{\dfrac{3-\tan^2A-1+3\tan^2A}{3-\tan^2A}}+\frac{1}{\dfrac{1-3\tan^2A-3+\tan^2A}{1-3\tan^2A}}$$

$$=\frac{3-\tan^2A}{2(1+\tan^2A)}+\frac{1-3\tan^2A}{-2(1+\tan^2A)}$$

$$=\frac{3-\tan^2A-1+3\tan^2A}{2(1+\tan^2A)}=\frac{2+2\tan^2A}{2(1+\tan^2A)}=1.$$

(5) $\cos 4A+\cos 4B=2\cos 2(A+B)\,.\,\cos 2(A-B)$

$$=2\,.\,\{1-2\sin^2(A+B)\}\,.\,\{1-2\sin^2(A-B)\}.$$

(6) $\tan(45^\circ+\theta)-\tan(45^\circ-\theta)=\dfrac{1+\tan\theta}{1-\tan\theta}-\dfrac{1-\tan\theta}{1+\tan\theta}=\dfrac{4\tan\theta}{1-\tan^2\theta}$

$$=\frac{\dfrac{4\sin\theta}{\cos\theta}}{1-\dfrac{\sin^2\theta}{\cos^2\theta}}=\frac{4\sin\theta\,.\,\cos\theta}{\cos^2\theta-\sin^2\theta}=\frac{2\sin 2\theta}{\cos 2\theta}=\frac{2\sin^2 2\theta}{\cos 2\theta\,.\,\sin 2\theta}$$

$$=\frac{2(1-\cos^2 2\theta)}{\cos 2\theta\,.\,\sin 2\theta}$$

$$=2\cdot\frac{\dfrac{1}{\cos 2\theta}-\dfrac{\cos^2 2\theta}{\cos 2\theta}}{\sin 2\theta}=2\cdot\frac{\sec 2\theta-\cos 2\theta}{\sin 2\theta}.$$

(7) $$\cot^2\theta - \tan^2\theta = \frac{\cos^4\theta - \sin^4\theta}{\cos^2\theta \,.\, \sin^2\theta} = \frac{(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)}{\cos^2\theta \,.\, \sin^2\theta}$$

$$= \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta \,.\, \sin^2\theta} = \frac{\cos 2\theta}{\cos^2\theta \,.\, \sin^2\theta} = \frac{4\cos 2\theta}{4\cos^2\theta \,.\, \sin^2\theta} = \frac{4\cos 2\theta}{\sin^2 2\theta}$$

$$= \frac{8\cos 2\theta}{2\sin^2 2\theta} = \frac{8\cos 2\theta}{1 - \cos 4\theta}.$$

(8)

$$2\sin A \,.\, \cos 2A = 2\sin A(1 - 2\sin^2 A) = 2\sin A - 4\sin^3 A = \sin 3A - \sin A.$$

(9) $$\frac{\cos nA - \cos(n+2)A}{\sin(n+2)A - \sin nA} = \frac{2\sin(n+1)A \,.\, \sin A}{2\cos(n+1)A \,.\, \sin A} = \tan(n+1)A.$$

(10)

$$\cos 9A + 3\cos 7A + 3\cos 5A + \cos 3A = \cos 9A + \cos 3A + 3(\cos 7A + \cos 5A)$$

$$= 2\cos 6A \,.\, \cos 3A + 6\cos 6A \,.\, \cos A$$

$$= 2\cos 6A(\cos 3A + 3\cos A)$$

$$= 2\cos 6A \,.\, 4\cos^3 A = 8\cos^3 A \,.\, \cos 6A.$$

(11) $$\frac{\operatorname{cosec} 2A - \cot 2A}{\operatorname{cosec} 2A + \cot 2A} = \frac{1 - \cos 2A}{1 + \cos 2A} = \frac{2\sin^2 A}{2\cos^2 A} = \tan^2 A.$$

(12) $$\frac{1 - \sin A}{1 + \cos A} = \frac{\sin^2\frac{A}{2} + \cos^2\frac{A}{2} - 2\sin\frac{A}{2} \cdot \cos\frac{A}{2}}{1 + 2\cos^2\frac{A}{2} - 1} = \frac{\left(\cos\frac{A}{2} - \sin\frac{A}{2}\right)^2}{2\cos^2\frac{A}{2}}$$

$$= \frac{1}{2}\left(\frac{\cos\frac{A}{2} - \sin\frac{A}{2}}{\cos\frac{A}{2}}\right)^2 = \frac{1}{2}\left(1 - \tan\frac{A}{2}\right)^2.$$

(13) $$\frac{\cos 3A - 2\cos A}{\sin 3A + 2\sin A} \cdot \tan A = \frac{4\cos^3 A - 3\cos A - 2\cos A}{3\sin A - 4\sin^3 A + 2\sin A} \cdot \frac{\sin A}{\cos A}$$

$$= \frac{4\cos^2 A - 3 - 2}{3 - 4\sin^2 A + 2} = \frac{2(2\cos^2 A - 1) - 3}{2(1 - 2\sin^2 A) + 3} = \frac{2\cos 2A - 3}{2\cos 2A + 3}.$$

(14) $\tan(45^\circ - A) + \tan(45^\circ + A) = \dfrac{1-\tan A}{1+\tan A} + \dfrac{1+\tan A}{1-\tan A}$

$$= \frac{2(1+\tan^2 A)}{1-\tan^2 A} = \frac{2\sec^2 A}{(\cos^2 A - \sin^2 A)\sec^2 A} = \frac{2}{\cos^2 A - \sin^2 A} = 2\sec 2A.$$

(15) $\cos 2a + \tan\dfrac{a}{2}\sin 2a = \cos 2a + \dfrac{\sin\frac{a}{2}}{\cos\frac{a}{2}} \cdot 2\sin a \,.\, \cos a$

$$= \cos 2a + 4\sin^2\frac{a}{2} \cdot \cos a = 2\cos^2 a - 1 + 4\sin^2\frac{a}{2} \cdot \cos a$$

$$= 2\cos a\left(\cos a + 2\sin^2\frac{a}{2}\right) - 1 = 2\cos a \,.\, 1 - 1 = 2\cos a - 1$$

$$= \cos a + \cos a - 1 = \cos a - 2\sin^2\frac{a}{2} = \cos a - \frac{2 \,.\, \sin^2\frac{a}{2} \cdot \cos\frac{a}{2}}{\cos\frac{a}{2}}$$

$$= \cos a - \tan\frac{a}{2} \cdot \sin a.$$

(16) $\cot^2 A - \tan^2 A = \dfrac{\cos^4 A - \sin^4 A}{\cos^2 A \,.\, \sin^2 A} = \dfrac{(\cos^2 A + \sin^2 A)(\cos^2 A - \sin^2 A)}{\cos^2 A \,.\, \sin^2 A}$

$$= \frac{\cos^2 A - \sin^2 A}{\cos^2 A \,.\, \sin^2 A} = \frac{4(\cos^2 A - \sin^2 A)}{4\cos^2 A \,.\, \sin^2 A} = \frac{4\cos 2A}{\sin^2 2A} = 4\cot 2A \,.\, \operatorname{cosec} 2A.$$

(17) $\operatorname{cosec} a \,.\, \cot a - \sec a \,.\, \tan a = \dfrac{\cos a}{\sin^2 a} - \dfrac{\sin a}{\cos^2 a} = \dfrac{\cos^3 a - \sin^3 a}{\sin^2 a \,.\, \cos^2 a}$

$$= \frac{4(\cos^3 a - \sin^3 a)}{4\sin^2 a \,.\, \cos^2 a} = \frac{4(\cos^3 a - \sin^3 a)}{\sin^2 2a} = 4\operatorname{cosec}^2 2a \,.\, (\cos^3 a - \sin^3 a).$$

(18) $\cot^2 a - \tan^2 a = \dfrac{\cos^2 a - \sin^2 a}{\cos^2 a \,.\, \sin^2 a} = \dfrac{4(\cos^2 a - \sin^2 a)}{4\cos^2 a \,.\, \sin^2 a} = \dfrac{4\cos 2a}{\sin^2 2a}.$

(19) $\operatorname{cosec}^2 b - \sec^2 b = \dfrac{1}{\sin^2 b} - \dfrac{1}{\cos^2 b} = \dfrac{\cos^2 b - \sin^2 b}{\sin^2 b \,.\, \cos^2 b} = \dfrac{4(\cos^2 b - \sin^2 b)}{4\sin^2 b \,.\, \cos^2 b}$

$$= \frac{4\cos 2b}{\sin^2 2b} = 4\cos 2b \,.\, \operatorname{cosec}^2 2b.$$

(20) $$\frac{2\,\mathrm{cosec}2A-\sec A}{2\,\mathrm{cosec}2A+\sec A}=\frac{2-\sec A\,.\,\sin 2A}{2+\sec A\,.\,\sin 2A}=\frac{2-2\sin A}{2+2\sin A}=\frac{1-\sin A}{1+\sin A}$$

$$=\frac{\sin^2\frac{A}{2}+\cos^2\frac{A}{2}-2\sin\frac{A}{2}\cdot\cos\frac{A}{2}}{\sin^2\frac{A}{2}+\cos^2\frac{A}{2}+2\sin\frac{A}{2}\cdot\cos\frac{A}{2}}=\left(\frac{\cos\frac{A}{2}-\sin\frac{A}{2}}{\cos\frac{A}{2}+\sin\frac{A}{2}}\right)^2$$

$$=\left(\frac{1-\tan\frac{A}{2}}{1+\tan\frac{A}{2}}\right)^2=\cot^2\left(45^\circ+\frac{A}{2}\right).$$

(21) $$\sin\left(\frac{5\pi}{2}+\theta\right)-\sin\left(\frac{3\pi}{2}-\theta\right)=2\cos 2\pi\,.\,\sin\left(\frac{\pi}{2}+\theta\right)$$

$$=2\cos 2\pi\,.\,\cos\theta=2\cos 2\pi\,.\,\sin\left(\frac{\pi}{2}-\theta\right)$$

$$=\sin\left(\frac{5\pi}{2}-\theta\right)-\sin\left(\frac{3\pi}{2}+\theta\right).\qquad\text{(Art. 122.)}$$

(22) $$\cot\left(\frac{\pi}{2}+\theta\right)-\tan\left(\frac{\pi}{2}+\theta\right)=\frac{\cos^2\left(\frac{\pi}{2}+\theta\right)-\sin^2\left(\frac{\pi}{2}+\theta\right)}{\sin\left(\frac{\pi}{2}+\theta\right).\cos\left(\frac{\pi}{2}+\theta\right)}$$

$$=\frac{\sin^2\theta-\cos^2\theta}{-\cos\theta\,.\,\sin\theta}=\frac{\cos^2\theta-\sin^2\theta}{\sin\theta\,.\,\cos\theta}=\frac{2\,.\,\cos 2\theta}{\sin 2\theta}=2\cot 2\theta.$$

(23) $$\frac{(\mathrm{cosec}\,a+\sec a)^2}{\mathrm{cosec}^2 a+\sec^2 a}=\frac{\left(\frac{\cos a+\sin a}{\sin a\,.\,\cos a}\right)^2}{\frac{1}{\sin^2 a\,.\,\cos^2 a}}=(\cos a+\sin a)^2=1+2\sin a\,.\,\cos a$$

$$=1+\sin 2a.$$

(24) $$\frac{\tan\theta}{\tan 2\theta-\tan\theta}=\frac{\tan\theta}{\frac{2\tan\theta}{1-\tan^2\theta}-\tan\theta}=\frac{1}{\frac{2}{1-\tan^2\theta}-1}=\frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$=\cos^2\theta-\sin^2\theta=\cos 2\theta.$$

(25) $$\frac{\tan 2\theta\,.\,\tan\theta}{\tan 2\theta-\tan\theta}=\frac{\frac{2\tan\theta}{1-\tan^2\theta}\cdot\tan\theta}{\frac{2\tan\theta}{1-\tan^2\theta}-\tan\theta}=\frac{\frac{2\tan\theta}{1-\tan^2\theta}}{\frac{2}{1-\tan^2\theta}-1}=\frac{2\tan\theta}{1+\tan^2\theta}$$

$$=\frac{2\tan\theta}{\sec^2\theta}=2\sin\theta\,.\,\cos\theta=\sin 2\theta.$$

(26) $$\frac{\sin^2\alpha - \sin^2\beta}{\sin\alpha.\cos\alpha - \sin\beta.\cos\beta} = \frac{\sin(\alpha+\beta).\sin(\alpha-\beta)}{\sin\alpha.\cos\alpha - \sin\beta.\cos\beta}$$

(Ex. XXVII. 1.)

$$= \frac{2\sin(\alpha+\beta).\sin(\alpha-\beta)}{2\sin\alpha.\cos\alpha - 2\sin\beta.\cos\beta} = \frac{2\sin(\alpha+\beta).\sin(\alpha-\beta)}{\sin2\alpha - \sin2\beta}$$

$$= \frac{2\sin(\alpha+\beta).\sin(\alpha-\beta)}{2\cos(\alpha+\beta).\sin(\alpha-\beta)} = \tan(\alpha+\beta).$$

(27) $4\sin A.\sin(60^\circ + A).\sin(60^\circ - A) = 4\sin A.(\sin^2 60^\circ - \sin^2 A).$

(Ex. XXVII. 1.)

$$= 4\sin A\left(\frac{3}{4} - \sin^2 A\right) = 3\sin A - 4\sin^3 A = \sin3A.$$

(28) $$\operatorname{cosec}2\theta + \cot4\theta + \operatorname{cosec}4\theta = \frac{1}{\sin2\theta} + \frac{\cos4\theta}{\sin4\theta} + \frac{1}{\sin4\theta}$$

$$= \frac{2\cos2\theta + \cos4\theta + 1}{2\sin2\theta.\cos2\theta} = \frac{2\cos2\theta + 2\cos^2 2\theta}{2\sin2\theta.\cos2\theta} = \frac{1+\cos2\theta}{\sin2\theta}$$

$$= \frac{2\cos^2\theta}{2\sin\theta.\cos\theta} = \frac{\cos\theta}{\sin\theta} = \cot\theta.$$

2. (1) $\sin2\theta + \sqrt{3}.\cos2\theta = 1,$

$$\sqrt{3}.\cos2\theta = 1 - \sin2\theta,$$
$$3.\cos^2 2\theta = 1 - 2\sin2\theta + \sin^2 2\theta,$$
$$3 - 3\sin^2 2\theta = 1 - 2\sin2\theta + \sin^2 2\theta.$$

Solving this quadratic, we obtain $\sin2\theta = -\frac{1}{2}$, or, 1;

$$\therefore 2\theta = -30^\circ, \text{ or, } 90^\circ;$$
$$\therefore \theta = -15^\circ, \text{ or, } 45^\circ.$$

(2) $$\sin^2 2\theta - \sin^2\theta = \sin^2\frac{\pi}{4},$$

$$4\sin^2\theta.\cos^2\theta - \sin^2\theta = \frac{1}{2},$$

$$4\sin^2\theta - 4\sin^4\theta - \sin^2\theta = \frac{1}{2}.$$

Solving this quadratic, we obtain $\sin^2\theta = \frac{1}{2}$, or, $\frac{1}{4}$;

$$\therefore \sin\theta = \frac{1}{\sqrt{2}}, \text{ or, } \frac{1}{2};$$
$$\therefore \theta = 45^\circ, \text{ or, } 30^\circ.$$

(3) $\sin 5x \cdot \cos 3x = \sin 9x \cdot \cos 7x$;

$\therefore \sin 8x + \sin 2x = \sin 16x + \sin 2x$;

$\therefore \sin 8x = \sin 16x$,

$\sin 8x = 2 \sin 8x \cdot \cos 8x$.

Hence $\sin 8x = 0$, or, $2 \cos 8x = 1$,

$\sin 8x = 0$, or, $\cos 8x = \frac{1}{2}$;

$\therefore x = 0°$, or, $8x = 60°$, and $\therefore x = 7\frac{1}{2}°$.

(4) $2 \sin^2 3\theta + \sin^2 6\theta = 2$,

$\sin^2 6\theta = 2(1 - \sin^2 3\theta)$,

$4 \sin^2 3\theta \cdot \cos^2 3\theta = 2 \cos^2 3\theta$,

$2 \sin 3\theta \cdot \cos 3\theta = \sqrt{2} \cos 3\theta$.

Hence $\cos 3\theta = 0$, or, $\sin 3\theta = \frac{1}{\sqrt{2}}$;

$\therefore 3\theta = 90°$, or, $3\theta = 45°$;

$\therefore \theta = 30°$, or, $15°$.

(5) $\cos 2A + \sin^2 A = \frac{3}{4}$

$1 - 2\sin^2 A + \sin^2 A = \frac{3}{4}$,

$\sin^2 A = \frac{1}{4}$, and $\therefore \sin A = \pm \frac{1}{2}$.

Hence $A = 30°$, or, $150°$.

(6) $\cos 3\theta - \cos 5\theta = \sin\theta$,

$2 \sin 4\theta \cdot \sin\theta = \sin\theta$.

Hence $\sin\theta = 0$, or, $\sin 4\theta = \frac{1}{2}$;

$\therefore \theta = 0°$, or, $4\theta = 30°$;

$\therefore \theta = 0°$, or, $\theta = 7\frac{1}{2}°$.

(7) $\sin 5\theta - \cos 3\theta = \sin\theta$,

$$\sin 5\theta - \sin\theta = \cos 3\theta,$$

$$2\cos 3\theta \,.\, \sin 2\theta = \cos 3\theta.$$

Hence $\cos 3\theta = 0$, or, $\sin 2\theta = \frac{1}{2}$;

$\therefore\ 3\theta = 90°$, or, $2\theta = 30°$,

$\therefore\ \theta = 30°$, or, $\theta = 15°$.

(8) $\tan 2a = 3\tan a$,

$$\frac{2\tan a}{1-\tan^2 a} = 3\tan a.$$

Hence $\tan a = 0$, and $\therefore\ a = 0°$,

or, $2 = 3 - 3\tan^2 a$,

$\tan^2 a = \frac{1}{3}$, or, $\tan a = \frac{1}{\sqrt{3}}$, or, $a = 30°$.

(9) $\sin 2\theta + \sin\theta = \cos 2\theta + \cos\theta$,

$$2\sin\frac{3\theta}{2}\cdot\cos\frac{\theta}{2} = 2\cos\frac{3\theta}{2}\cdot\cos\frac{\theta}{2}.$$

$\therefore\ \cos\frac{\theta}{2} = 0$, or, $\frac{\theta}{2} = 90°$, or, $\theta = 180°$;

or, $\sin\frac{3\theta}{2} = \cos\frac{3\theta}{2}$, or, $\tan\frac{3\theta}{2} = 1$, or, $\frac{3\theta}{2} = 45°$, or, $\theta = 30°$.

(10) $\sin 7a - \sin a = \sin 3a$,

$$2\cos 4a \,.\, \sin 3a = \sin 3a.$$

Hence $\sin 3a = 0$, or, $\left.\begin{array}{l} 3a = 0°, \text{ or, } a = 0° \\ 3a = 180°, \text{ or, } a = 60° \end{array}\right\}$,

or, $2\cos 4a = 1$, or, $4a = 60°$, or, $a = 15°$.

(11) $\operatorname{cosec}^2\theta - \sec^2\theta = 2\operatorname{cosec}^2\theta \div 3$,

$\frac{\operatorname{cosec}^2\theta}{3} = \sec^2\theta$, or, $\cos^2\theta = 3\sin^2\theta$;

$\therefore\ 4\sin^2\theta = 1$, or, $\sin\theta = \frac{1}{2}$, and $\therefore\ \theta = 30°$.

(12) $\sin 6\theta = \sin 4\theta - \sin 2\theta,$

$\sin 6\theta + \sin 2\theta = \sin 4\theta,$

$2 \sin 4\theta . \cos 2\theta = \sin 4\theta.$

Hence $\sin 4\theta = 0$, or, $4\theta = 0°$, or, $\theta = 0°$,

or, $2 \cos 2\theta = 1$, or, $\cos 2\theta = \frac{1}{2}$, or, $\theta = 30°$.

EXAMPLES—XXXVI. (p. 106).

1. (1) $\sin 36° = 2 \sin 18° . \cos 18° = 2 \cdot \frac{\sqrt{5}-1}{4} \cdot \frac{\sqrt{(10+2\sqrt{5})}}{4}$

$$= \frac{2\sqrt{(40-8\sqrt{5})}}{16} = \frac{\sqrt{(10-2\sqrt{5})}}{4}.$$

(2) $\cos 36° = 1 - 2\sin^2 18° = 1 - 2 . \left(\frac{\sqrt{5}-1}{4}\right)^2 = 1 - \frac{6-2\sqrt{5}}{8} = \frac{1+\sqrt{5}}{4}.$

(3) $\sin 54° = \cos 36° = \frac{1+\sqrt{5}}{4}.$

(4) $\cos 54° = \sin 36° = \frac{\sqrt{(10-2\sqrt{5})}}{4}.$

(5) $\sin 72° = \cos 18° = \sqrt{(1-\sin^2 18°)} = \frac{\sqrt{(10+2\sqrt{5})}}{4}.$

(6) $\tan 72° = \frac{\sin 72°}{\cos 72°} = \frac{\cos 18°}{\sin 18°} = \frac{\sqrt{(10+2\sqrt{5})}}{4} \div \frac{\sqrt{5}-1}{4} = \frac{\sqrt{(10+2\sqrt{5})}}{\sqrt{5}-1}.$

(7) $\sin 90° = \sin(18° + 72°) = \sin 18° . \cos 72° + \cos 18° . \sin 72$

$$= \sin 18° . \sin 18° + \cos 18° . \cos 18°$$

$$= \left(\frac{\sqrt{5}-1}{4}\right)^2 + \left(\frac{\sqrt{(10+2\sqrt{5})}}{4}\right)^2$$

$$= \frac{6-2\sqrt{5}+10+2\sqrt{5}}{16} = \frac{16}{16} = 1.$$

(8) $\cos 90° = \cos(18° + 72°) = \cos 18° . \cos 72° - \sin 18° . \sin 72°$

$$= \cos 18° . \cos 72° - \cos 72° . \cos 18° = 0.$$

2. $\sin(36^\circ+A)+\sin(72^\circ-A)-\sin(36^\circ-A)-\sin(72^\circ+A)$

$=\{\sin(36^\circ+A)-\sin(36^\circ-A)\}-\{\sin(72^\circ+A)-\sin(72^\circ-A)\}$

$=2\cos36^\circ.\sin A-2\cos72^\circ.\sin A$

$=\sin A\{2\cos36^\circ-2\cos72^\circ\}=\sin A\left\{\dfrac{1+\sqrt{5}}{2}-\dfrac{\sqrt{5}-1}{2}\right\}=\sin A.$

Also,

$\{\sin(54^\circ+A)+\sin(54^\circ-A)\}-\{\sin(18^\circ+A)+\sin(18^\circ-A)\}$

$=2\sin54^\circ.\cos A-2\sin18^\circ.\cos A$

$=\cos A\{2\sin54^\circ-2\sin18^\circ\}=\cos A\left\{\dfrac{1+\sqrt{5}}{2}-\dfrac{\sqrt{5}-1}{2}\right\}=\cos A.$

EXAMPLES—XXXVII (p. 110).

(1) At $7\frac{1}{2}^\circ$ the cosine is greater than the sine, and both are positive;

$$\therefore\ \cos\frac{A}{2}+\sin\frac{A}{2}=+\sqrt{1+\sin A},$$

$$\cos\frac{A}{2}-\sin\frac{A}{2}=+\sqrt{1-\sin A}.$$

(2) At 150° the cosine (negative) is greater than the sine (positive);

$$\therefore\ \cos\frac{A}{2}+\sin\frac{A}{2}=-\sqrt{1+\sin A},$$

$$\cos\frac{A}{2}-\sin\frac{A}{2}=-\sqrt{1-\sin A}.$$

(3) $\cos189^\circ+\sin189^\circ=-\sqrt{1+\sin378^\circ}$,

$\cos189^\circ-\sin189^\circ=-\sqrt{1-\sin378^\circ}$;

$$\therefore\ \cos189^\circ=-\frac{1}{2}\cdot\left\{\sqrt{1+\frac{\sqrt{5}-1}{4}}+\sqrt{1-\frac{\sqrt{5}-1}{4}}\right\}$$

$$=-\frac{1}{2}\cdot\left\{\frac{\sqrt{3+\sqrt{5}}}{2}+\frac{\sqrt{5-\sqrt{5}}}{2}\right\}$$

$$=-\frac{1}{4}\left\{\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}}\right\},$$

$$\text{and } \sin 189^\circ = \frac{1}{2}\left\{\sqrt{1-\frac{\sqrt{5}-1}{4}} - \sqrt{1+\frac{\sqrt{5}-1}{4}}\right\}$$

$$= \frac{1}{4}\left\{\sqrt{5-\sqrt{5}} - \sqrt{3+\sqrt{5}}\right\}.$$

(4) $$2\sin 9^\circ.\,44'.\,30'' = \sqrt{1+\frac{1}{3}} - \sqrt{1-\frac{1}{3}}$$

$$= \sqrt{\frac{4}{3}} - \sqrt{\frac{2}{3}} = \frac{2-\sqrt{2}}{\sqrt{3}};$$

$$\therefore \sin 9^\circ.\,44'.\,30'' = \frac{2-\sqrt{2}}{2\sqrt{3}}.$$

(5) $$\cos 157^\circ.\,30' = -\sqrt{\frac{1+\cos 315^\circ}{2}} = -\sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}} = -\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

$$= -\sqrt{\frac{2+\sqrt{2}}{4}} = -\frac{\sqrt{2+\sqrt{2}}}{2}.$$

EXAMPLES—XXXVIII. (p. 111).

(1) $$\sin A = \frac{3}{5} \text{ and } \sin B = \frac{4}{5},$$

$$\cos A = \frac{4}{5} \text{ and } \cos B = \frac{3}{5};$$

$$\therefore \sin(A+B) = \frac{3}{5}\cdot\frac{3}{5} + \frac{4}{5}\cdot\frac{4}{5} = \frac{25}{25} = 1;$$

$$\therefore A+B = 90^\circ.$$

(2) $$\tan A = \frac{1}{7};\ \tan B = \frac{1}{3},$$

$$\tan 2B = \frac{2\tan B}{1-\tan^2 B} = \frac{2}{3} \div \left(1-\frac{1}{9}\right) = \frac{2\times 9}{3\times 8} = \frac{3}{4};$$

$$\therefore \tan(A+2B) = \frac{\tan A + \tan 2B}{1-\tan A\,.\,\tan 2B} = \frac{\frac{1}{7}+\frac{3}{4}}{1-\frac{1}{7}\cdot\frac{3}{4}} = 1;$$

$$\therefore A+2B = 45^\circ.$$

(3) Let $\sin A = \frac{1}{\sqrt{5}}$ and $\cot B = 3$.

Then $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$;

$$\therefore \tan(A+B) = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}} = 1;$$

$$\therefore A+B=45°,$$

$$\text{that is } \sin^{-1}\frac{1}{\sqrt{5}} + \cot^{-1}3 = 45°.$$

(4) Let A, B, C, D be the four angles whose tangents are

$$\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{8}.$$

Then $\tan\{(A+B)+(C+D)\}$

$$= \frac{\tan(A+B)+\tan(C+D)}{1-\tan(A+B)\,.\,\tan(C+D)}$$

$$= \left(\frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{15}} + \frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\cdot\frac{1}{8}}\right) \div \left(1-\frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{15}}\cdot\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\cdot\frac{1}{8}}\right)$$

$$= \left(\frac{4}{7}+\frac{3}{11}\right) \div \left(1-\frac{12}{77}\right) = 1;$$

$$\therefore A+B+C+D=45°.$$

(5) Let $\cot A = \frac{3}{4}$ and $\cot B = \frac{1}{7}$.

Then $\tan A = \frac{4}{3}$ and $\tan B = 7$;

$$\therefore \tan(A+B) = \frac{\frac{4}{3}+7}{1-\frac{28}{3}} = -1;$$

$$\therefore A+B=135°, \text{ or, } \cot^{-1}\frac{3}{4}+\cot^{-1}\frac{1}{7}=135°.$$

(6) $$\text{Let } \tan A=\frac{3}{5} \text{ and } \tan B=\frac{3}{7}.$$

$$\text{Then } \tan(A+B)=\frac{\frac{3}{5}+\frac{3}{7}}{1-\frac{9}{35}}=\frac{18}{13};$$

$$\therefore \cot(A+B)=\frac{13}{18}, \text{ or, } A+B=\cot^{-1}\frac{13}{18}.$$

(7) $$\text{Let } \tan A=x \text{ and } \tan B=y.$$

$$\text{Then } \tan(A-B)=\frac{x-y}{1+xy};$$

$$\therefore \tan^{-1}x-\tan^{-1}y=\tan^{-1}\frac{x-y}{1+xy}.$$

(8) Let $\sin A=x$ and $\cos B=x$.

Then $\cos A=\sqrt{1-x^2}$ and $\sin B=\sqrt{1-x^2}$;

$$\therefore \sin(A+B)=x.x+\sqrt{1-x^2}.\sqrt{1-x^2}$$
$$=x^2+1-x^2=1;$$
$$\therefore A+B=90^\circ, \text{ or, } \sin^{-1}x+\cos^{-1}x=90^\circ.$$

(9) $$\text{Let } \sin A=\frac{4}{5},\ \sin B=\frac{5}{13},\ \sin C=\frac{16}{65};$$

$$\therefore \cos A=\frac{3}{5},\ \cos B=\frac{12}{13},\ \cos C=\frac{63}{65}.$$

Then $\sin(A+B+C)=\sin(A+B).\cos C+\cos(A+B).\sin C$

$$=(\sin A.\cos B+\cos A.\sin B)\frac{63}{65}+(\cos A.\cos B-\sin A.\sin B)\frac{16}{65}$$

$$=\left(\frac{4}{5}\cdot\frac{12}{13}+\frac{3}{5}\cdot\frac{5}{13}\right)\cdot\frac{63}{65}+\left(\frac{3}{5}\cdot\frac{12}{13}-\frac{4}{5}\cdot\frac{5}{13}\right)\frac{16}{65}$$

$$=\frac{63}{65}\cdot\frac{63}{65}+\frac{16}{65}\cdot\frac{16}{65}=\frac{4225}{4225}=1.$$

$$\therefore A+B+C=90^\circ, \text{ or, } \sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}=\frac{\pi}{2}.$$

(10) Let $\tan A = \frac{1}{5}$, and $\tan B = \frac{1}{239}$.

$$\text{Then } \tan(4A - B) = \frac{\tan 4A - \tan B}{1 + \tan 4A \,.\, \tan B}$$

$$= \frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \cdot \frac{1}{239}} = 1\,;$$

$$\therefore\ 4A - B = 45^\circ,\ \text{or},\ 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = \frac{\pi}{4}.$$

EXAMPLES—XXXIX. (p. 120).

(1)
$$\begin{array}{r} \bar{3}\cdot1651553 \\ \bar{4}\cdot7505855 \\ 6\cdot6879746 \\ \bar{2}\cdot6150026 \\ \hline \bar{1}\cdot2187180 \end{array}$$

(2)
$$\begin{array}{r} \bar{4}\cdot6843785 \\ \bar{5}\cdot6650657 \\ 3\cdot8905196 \\ \bar{3}\cdot4675284 \\ \hline \bar{7}\cdot7074922 \end{array}$$

(3)
$$\begin{array}{r} 2\cdot5324716 \\ 3\cdot6650657 \\ \bar{5}\cdot8905196 \\ \cdot3156215 \\ \hline 2\cdot4036784 \end{array}$$

(4)
$$\begin{array}{r} 2\cdot483269 \\ \bar{3}\cdot742891 \\ \hline 4\cdot740378 \end{array}$$

(5)
$$\begin{array}{r} \bar{2}\cdot352678 \\ \bar{5}\cdot428619 \\ \hline 2\cdot924059 \end{array}$$

(6)
$$\begin{array}{r} \bar{5}\cdot349162 \\ \bar{3}\cdot624329 \\ \hline \bar{3}\cdot724833 \end{array}$$

(7)
$$\begin{array}{r} \bar{2}\cdot4596721 \\ 3 \\ \hline \bar{5}\cdot3790163 \end{array}$$

(8)
$$\begin{array}{r} \bar{7}\cdot429683 \\ 6 \\ \hline \overline{40}\cdot578098 \end{array}$$

(9)
$$\begin{array}{r} \bar{9}\cdot2843617 \\ 7 \\ \hline \overline{62}\cdot9905319 \end{array}$$

(10)
$$\begin{array}{r|l} 3 & \bar{6}\cdot3725409 \\ \hline & \bar{2}\cdot1241803 \end{array}$$

(11)
$$\begin{array}{r|l} 6 & \overline{14}\cdot432962 \\ \hline & \bar{3}\cdot738827 \end{array}$$

(12)
$$\begin{array}{r|l} 9 & \bar{4}\cdot53627188 \\ \hline & \bar{1}\cdot61514132 \end{array}$$

EXAMPLES—XL. (p. 123).

1. $\log 128 = \log 2^7 = 7\log 2 = 2{\cdot}1072100$

$$\log 125 = \log\frac{1000}{8} = \log 1000 - \log 8 = 3 - \log 2^3$$
$$= 3 - 3\log 2 = 3 - {\cdot}90309000 = 2{\cdot}0969100$$
$$\log 2500 = \log\frac{10000}{4} = \log 10000 - \log 4 = 4 - 2\log 2$$
$$= 4 - {\cdot}6020600 = 3{\cdot}3979400.$$

2. $\log 50 = \log\frac{100}{2} = \log 100 - \log 2 = 2 - {\cdot}3010300 = 1{\cdot}6989700$

$$\log {\cdot}005 = \log\frac{5}{1000} = \log 10 - \log 2 - 3 = -\log 2 - 2 = \bar{3}{\cdot}6989700$$
$$\log 196 = \log(49\times 4) = 2\log 7 + 2\log 2 = 2{\cdot}2922560.$$

3. $\log 6 = \log 3 + \log 2 = {\cdot}7781513$

$$\log 27 = 3\log 3 = 1{\cdot}4313639$$
$$\log 54 = \log(27\times 2) = 3\log 3 + \log 2 = 1{\cdot}7323939$$
$$\log 576 = \log(9\times 64) = 2\log 3 + 6\log 2 = 2{\cdot}7604226.$$

4. $\log 60 = \log(2\times 3\times 10) = \log 2 + \log 3 + \log 10 = 1{\cdot}7781513$

$$\log {\cdot}03 = \log\frac{3}{100} = \log 3 - 2 = {\cdot}4771213 - 2 = \bar{2}{\cdot}4771213$$
$$\log 1{\cdot}05 = \log\frac{105}{100} = \log\frac{21}{20} = \log 3 + \log 7 - \log 2 - 1 = {\cdot}0211893$$
$$\log {\cdot}0000432 = \log\frac{16\times 27}{10000000} = 4\log 2 + 3\log 3 - 7 = \bar{5}{\cdot}6354839$$

5. $\log {\cdot}00075 = \log 75 - 5 = \log 3 + \log 25 - 5 = \log\left(\frac{18}{2}\right)^{\frac{1}{2}} + \log 25 - 5$

$$= \frac{1}{2}\left\{\log 18 - \log 2\right\} + \log 100 - \log 4 - 5$$
$$= \frac{1}{2}\left\{1{\cdot}2552725 - {\cdot}3010300\right\} + 2 - {\cdot}6020600 - 5$$
$$= {\cdot}4771213 - {\cdot}6020600 - 3 = \bar{4}{\cdot}8750613.$$

$$\text{Log } 31{\cdot}5 = \log(21 \times 3 \times 5) - 1 = \log 21 + \log 3 + 1 - \log 2 - 1$$

$$= \log 21 + \frac{1}{2}\left(\log 18 - \log 2\right) - \log 2$$

$$= 1{\cdot}3222193 + {\cdot}4771212 - {\cdot}3010300 = 1{\cdot}4983105.$$

6. $$\log 2 = \log\frac{10}{5} = 1 - \log 5 = {\cdot}3010300.$$

$$\log {\cdot}064 = \log\frac{2^6}{1000} = 6\log 2 - 3 = 6 - 6\log 5 - 3 = \bar{2}{\cdot}8061800$$

$$\log\left\{\frac{2^{60}}{5^{20}}\right\}^{\frac{1}{14}} = \frac{1}{14}\left(60\log 2 - 20\log 5\right)$$

$$= \frac{1}{7}\left(30 - 30\log 5 - 10\log 5\right) = \frac{1}{7}\left(30 - 27{\cdot}9588000\right)$$

$$= \frac{1}{7}\left(2{\cdot}0412000\right) = {\cdot}2916000.$$

7. $$\log 5 = \log\frac{10}{2} = 1 - {\cdot}3010300 = {\cdot}6989700,$$

$$\log {\cdot}125 = \log\frac{5^3}{1000} = 3\log 5 - 3 = 2{\cdot}0969100 - 3 = \bar{1}{\cdot}0969100$$

$$\log\left(\frac{5^{90}}{2^{40}}\right)^{\frac{1}{15}} = \log 5^{\frac{90}{15}} - \log 2^{\frac{40}{15}} = \log 5^6 - \log 2^{\frac{8}{3}}$$

$$= 6\log 5 - \frac{8}{3}\log 2 = 6\left(\log 10 - \log 2\right) - \frac{8}{3}\log 2$$

$$= 4{\cdot}1938200 - {\cdot}8027467 = 3{\cdot}3910733.$$

8. $$\left.\begin{aligned} {\cdot}01 &= \frac{1}{100} = \frac{1}{10^2} = 10^{-2} \\ 1 &= 10^0 \\ 100 &= 10^2 \end{aligned}\right\} \therefore \text{ the logarithms are } -2,\ 0,\ 2\,;$$

$$\left.\begin{aligned} {\cdot}01 &= ({\cdot}01)^1 \\ 1 &= ({\cdot}01)^0 \\ 100 &= \frac{1}{{\cdot}01} = ({\cdot}01)^{-1} \end{aligned}\right\} \therefore \text{ the logarithms are } 1,\ 0,\ -1.$$

9. 1593 is greater than 10^3 and less than 10^4; characteristic 3.
1593 is greater than 12^2 and less than 12^3; characteristic 2.

10. $\frac{4^{3y}}{2^{4y}}=8$; $\frac{2^{6y}}{2^{4y}}=2^3$; $2^{2y}=2^3$; $2y=3$.

$$\text{Hence } y=\frac{3}{2} \text{ and } x=\frac{9}{2}.$$

11. (*a*) $\log 2=\frac{1}{2}\log 4=\cdot 3010300,$

$$\log 25=\log 100-\log 4=2-\cdot 6020600=1\cdot 3979400$$

$$\log 83\cdot 2=\log(80\times 1\cdot 04)=\frac{3}{2}\log 4+\log 10+\log 1\cdot 04$$

$$=\cdot 9030900+1+\cdot 0170333=1\cdot 9201233$$

$$\log(\cdot 625)^{\frac{1}{100}}=\frac{1}{100}\left\{\log 625-\log 1000\right\}=\frac{1}{100}\left\{2\log 25-3\right\}$$

$$=\frac{1}{100}\left\{2\log 100-2\log 4-3\right\}=\frac{1}{100}\left\{4-1\cdot 2041200-3\right\}$$

$$=-\cdot 0020412=\bar{1}\cdot 9979588.$$

(*b*) $\log(1\cdot 04)^{6000}=6000\log 1\cdot 04=6000\times\cdot 0170333$

$=102\cdot 1998000$; $\therefore$ number of digits is 103.

12. (*a*) $\log 5=\frac{1}{2}\log 25=\cdot 6989700$

$$\log 4=2-\log 25=\cdot 6020600$$

$$\log 51\cdot 5=\log 5+\log 10\cdot 3=\cdot 6989700+1\cdot 0128372=1\cdot 7118072$$

$$\log(\cdot 064)^{\frac{1}{100}}=\frac{1}{100}\left\{\log 64-\log 1000\right\}=\frac{1}{100}\left\{3\log 4-3\right\}$$

$$=\frac{1}{100}\left\{1\cdot 8061800-3\right\}=-\cdot 0119382=\bar{1}\cdot 9880618.$$

(*b*) $\log(1\cdot 03)^{600}=600\log 1\cdot 03=600\times\cdot 0128372$

$=7\cdot 7023200$; $\therefore$ number of digits is 8.

13. $\log 7623 = \log(9 \times 121 \times 7) = 2\log 3 + 2\log 11 + \log 7$

$$= \cdot 9542426 + 2\cdot 0827854 + \cdot 8450980 = 3\cdot 8821260$$

$$\log\frac{77}{300} = \log 7 + \log 11 - \log 3 - \log 100$$

$$= \cdot 8450980 + 1\cdot 0413927 - \cdot 4771213 - 2 = \bar{1}\cdot 4093694$$

$$\log\frac{3}{539} = \log 3 - \log 11 - 2\log 7$$

$$= \cdot 4771213 - 1\cdot 0413927 - 1\cdot 6901960 = \bar{3}\cdot 7455326.$$

14. (1)

$$x\log 4096 = \log 8 - x\log 64$$
$$4x\log 8 = \log 8 - 2x\log 8$$
$$4x = 1 - 2x;\ 6x = 1;\ x = \frac{1}{6}.$$

(2)

$$(2\cdot 5)^x = 6\cdot 25 = (2\cdot 5)^2;\ \therefore\ x = 2.$$

(3)

$$(ab)^x = m;\ x\log(ab) = \log m;$$
$$\therefore\ x = \frac{\log m}{\log a + \log b}.$$

(4)

$$x(m\log a + 2\log b) = \log c;$$
$$\therefore\ x = \frac{\log c}{m\log a + 2\log b}.$$

(5)

$$3x\log a + (4 - x)\log b = (2x - 1)\log c$$
$$x(3\log a - \log b - 2\log c) = -4\log b - \log c;$$
$$\therefore\ x = \frac{4\log b + \log c}{2\log c + \log b - 3\log a}.$$

(6)

$$x(\log a + m\log b) = \log c - 3x\log c$$
$$x(\log a + m\log b + 3\log c) = \log c;$$
$$\therefore\ x = \frac{\log c}{\log a + m\log b + 3\log c}.$$

EXAMPLES—XLI. (p. 127).

(1) $\log 525030 = 5{\cdot}7201841$

$\log 525020 = 5{\cdot}7201758$

Difference for 10 = $\cdot 0000083$

$\therefore$ 10 : 5 = $\cdot 0000083$: what we must add;

$\therefore$ we must add $\cdot 0000041$;

$\therefore \log 52502{\cdot}5 = 4{\cdot}7201799.$

(2) $\log 300430 = 5{\cdot}4777433$

$\log 300420 = 5{\cdot}4777288$

Difference for 10 = $\cdot 0000145$

$\therefore$ 10 : 5 = $\cdot 0000145$: what we must add;

$\therefore$ we must add $\cdot 0000072$;

$\therefore \log 300{\cdot}425 = 2{\cdot}4777360.$

(3) $\log 32026000 = 7{\cdot}5055027$

$\log 32025000 = 7{\cdot}5054891$

Difference for 1000 = $\cdot 0000136$

$\therefore$ 1000 : 613 = $\cdot 0000136$: what we must add;

$\therefore$ we must add $\cdot 0000083$;

$\therefore \log 32{\cdot}025613 = 1{\cdot}5054974.$

(4) $\log 236610 = 5{\cdot}3740331$

$\log 236600 = 5{\cdot}3740147$

Difference for 10 = $\cdot 0000184$

$\therefore$ 10 : 1 = $\cdot 0000184$: what we must add;

$\therefore$ we must add $\cdot 0000018$;

$\therefore \log 236{\cdot}601 = 2{\cdot}3740165.$

(5)
$$\log 675030 = 5{\cdot}8293231$$
$$\log 675020 = 5{\cdot}8293166$$
$$\text{Difference for } 10 = {\cdot}0000065$$
$\therefore$ 10 : 1 = ·0000065 : what we must add ;

$\therefore$ we must add ·0000007 (see end of Art. 162) ;

$\therefore$ $\log 67{\cdot}5021 = 1{\cdot}8293173.$

(6)
$$\log 7333600 = 6{\cdot}8653172$$
$$\log 7333500 = 6{\cdot}8653113$$
$$\text{Difference for } 100 = {\cdot}0000059$$
$\therefore$ 100 : 33 = ·0000059 : what we must add ;

$\therefore$ we must add ·0000019 ;

$\therefore$ $\log {\cdot}007333533 = \bar{3}{\cdot}8653132.$

(7)
$$\log 6593200 = 6{\cdot}8190962$$
$$\log 6593100 = 6{\cdot}8190897$$
$$\text{Difference for } 100 = {\cdot}0000065$$
$\therefore$ 100 : 71 = ·0000065 : what we must add ;

$\therefore$ we must add ·0000046 ;

$\therefore$ $\log {\cdot}000006593171 = \bar{6}{\cdot}8190943.$

(8)
$$\log 340780 = 5{\cdot}5324741$$
$$\log 340770 = 5{\cdot}5324614$$
$$\text{Difference for } 10 = {\cdot}0000127$$
$\therefore$ 10 : 8 = ·0000127 : what we must add ;

$\therefore$ we must add ·0000102 ;

$\therefore$ $\log 3407{\cdot}78 = 3{\cdot}5324716.$

(9)
$$\log 390980 = 5{\cdot}5921545$$
$$\log 390970 = 5{\cdot}5921434$$
$$\text{Difference for } 10 = {\cdot}0000111$$
$\therefore$ 10 : 4 = ·0000111 : what we must add ;

$\therefore$ we must add ·0000044 ;

$\therefore$ $\log 390974 = 5{\cdot}5921478.$

(10)
$$\log 2582000 = 6{\cdot}4119562$$
$$\log 2581900 = \underline{6{\cdot}4119394}$$
$$\text{Difference for } 100 = \ {\cdot}0000168$$
$\therefore$ 100 : 26 := ·0000168 : what we must add ;

$\therefore$ we must add ·0000044 ;

$\therefore$ log 2·581926 = ·4119438.

EXAMPLES—XLII. (p. 129).

(1)
$$\log 12955 = 4{\cdot}1124374$$
$$\log 12954 = \underline{4{\cdot}1124039}$$
$$\text{Difference for } 1 = \ {\cdot}0000335$$
$\therefore$ ·0000335 : ·0000271 = 1 : what has to be added ;

$\therefore$ we must add ·8 ;

$\therefore$ 4·112431 is the logarithm of 12954·8.

(2)
$$\log 46246 = 4{\cdot}6650742$$
$$\log 46245 = \underline{4{\cdot}6650648}$$
$$\text{Difference for } 1 = \ {\cdot}0000094$$
$\therefore$ ·0000094 : ·0000009 = 1 : what has to be added ;

$\therefore$ we must add ·095 ;

$\therefore$ 3·6650657 is the logarithm of 4624·5095.

(3)
$$\log 34573 = 4{\cdot}5387371$$
$$\log 34572 = \underline{4{\cdot}5387245}$$
$$\text{Difference for } 1 = \ {\cdot}0000126$$
$\therefore$ ·0000126 : 0000114 = 1 : what we must add ;

$\therefore$ we must add ·9047 . . ., or, ·91 ;

$\therefore$ 2·5387359 is the logarithm of 345·7291.

(4) log 39376 = 4·5952316
log 39375 = 4·5952206

Difference for 1 = ·0000110

∴ ·0000110 : ·0000076 = 1 : what we must add;
∴ we must add ·69;
∴ 5·5952282 is the logarithm of 393756·9.

(5) log 37160 = 4·5700757
log 37159 = 4·5700640

Difference for 1 = ·0000117

∴ ·0000117 : ·0000062 = 1 : what we must add;
∴ we must add ·529, or, ·53;
∴ 3·5700702 is the logarithm of 3715·953.

(6) log 96462 = 4·9843563
log 96461 = 4·9843518

Difference for 1 = ·0000045

∴ ·0000045 : ·0000024 = 1 : what we must add;
∴ we must add $\cdot 5\dot{3}$;
∴ $\bar{3}\cdot 9843542$ is the logarithm of $\cdot 00964615\dot{3}$.

(7) log 25726 = 4·4103723
log 25725 = 4·4103554

Difference for 1 = ·0000169

∴ ·0000169 : ·0000166 = 1 : what must be added;
∴ we must add ·982;
∴ $\bar{7}\cdot 4103720$ is the logarithm of ·00000025725982.

(8) log 60196 = 4·7795604
log 60195 = 4·7795532

Difference for 1 = ·0000072

∴ ·0000072 : ·0000029 = 1 : what must be added
∴ we must add ·4027, or, ·403;
∴ 2·7795561 is the logarithm of 601·95403.

(9) log 10906 = 4·0376655

log 10905 = 4·0376257

Difference for 1 = ·0000398

∴ ·0000398 : ·0000114 = 1 : what must be added;

∴ we must add ·286;

∴ 3·0376371 is the logarithm of 1090·5286.

(10) log 26202 = 4·4183344

log 26201 = 4·4183179

Difference for 1 = ·0000165

∴ ·0000165 : ·0000135 = 1 : what must be added;

∴ we must add ·818;

∴ 2·4183314 is the logarithm of 262·01818.

EXAMPLES—XLIII. (p. 132).

(1) sin42°. 16′ = ·6725821

sin42°. 15′ = ·6723668

Difference for 1′ = ·0002153

∴ 60″ : 16″ = ·0002153 : what we must *add*;

∴ we must add ·0000574;

∴ sin42°. 15′. 16″ = ·6724242.

(2) sin72°. 15′ = ·9523958

sin72°. 14′ = ·9523071

Difference for 1′ = ·0000887

∴ 60″ : 6″ = ·0000887 : what we must *add*;

∴ we must add ·0000088;

∴ sin72°. 14′. 6″ = ·9523159.

(3)

$$\sin 54°.\,36' = \cdot 8151278$$
$$\sin 54°.\,35' = \cdot 8149593$$
$$\text{Difference for } 1' = \cdot 0001685$$
$$\therefore\ 60'' : 45'' = \cdot 0001685 : \text{what we must } \textit{add};$$
$$\therefore\ \text{we must add } \cdot 0001263;$$
$$\therefore\ \sin 54°.\,35'.\,45'' = \cdot 8150856.$$

(4)

$$\sin 87°.\,27' = \cdot 9990098$$
$$\sin 87°.\,26' = \cdot 9989968$$
$$\text{Difference for } 1' = \cdot 0000130$$
$$\therefore\ 60'' : 15'' = \cdot 0000130 : \text{what we must } \textit{add};$$
$$\therefore\ \text{we must add } \cdot 0000032;$$
$$\therefore\ \sin 87°.\,26'.\,15'' = \cdot 9990000.$$

(5)

$$\sin 43°.\,15' = \cdot 6851830$$
$$\sin 43°.\,14' = \cdot 6849711$$
$$\text{Difference for } 1' = \cdot 0002119$$
$$\therefore\ 60'' : 20'' = \cdot 0002119 : \text{what we must } \textit{add};$$
$$\therefore\ \text{we must add } \cdot 0000706;$$
$$\therefore\ \sin 43°.\,14'.\,20'' = \cdot 6850417.$$

(6)

$$\cos 41°.\,13' = \cdot 7522233$$
$$\cos 41°.\,14' = \cdot 7520316$$
$$\text{Difference for } 1' = \cdot 0001917$$
$$\therefore\ 60'' : 26'' = \cdot 0001917 : \text{what we must } \textit{subtract};$$
$$\therefore\ \text{we must subtract } \cdot 0000830;$$
$$\therefore\ \cos 41°.\,13'.\,26'' = \cdot 7521403.$$

(7)

$$\tan 1°.\,23' = \cdot 0241484$$
$$\tan 1°.\,22' = \cdot 0238573$$
$$\text{Difference for } 1' = \cdot 0002911$$
$$\therefore\ 60'' : 30'' = \cdot 0002911 : \text{what we must } \textit{add};$$
$$\therefore\ \text{we must add } \cdot 0001455;$$
$$\therefore\ \tan 1°.\,22'.\,30'' = \cdot 0240028.$$

(8) $$\cot 35°.\,6' = 1\cdot4228561$$
$$\cot 35°.\,7' = 1\cdot4219766$$
$$\text{Difference for } 1' = \cdot0008795$$

$\therefore$ $60'' : 23'' = \cdot0008795$: what we must subtract;

$\therefore$ we must subtract $\cdot0003371$;

$\therefore$ $\cot 35°.\,6'.\,23'' = 1\cdot4225190$.

(9) $$\sin 67°.\,23' = \cdot9230984$$
$$\sin 67°.\,22' = \cdot9229865$$
$$\text{Difference for } 1' = \cdot0001119$$

$\therefore$ $60'' : 48''\cdot5 = \cdot0001119$: what we must *add*;

$\therefore$ we must add $\cdot0000904$;

$\therefore$ $\sin 67°.\,22'.\,48''\cdot5 = \cdot9230769$.

(10) $$\cos 34°.\,12' = \cdot8270806$$
$$\cos 34°.\,13' = \cdot8269170$$
$$\text{Difference for } 1' = \cdot0001636$$

$\therefore$ $60'' : 19''\cdot6 = \cdot0001636$: what we must *subtract*;

$\therefore$ we must subtract $\cdot0000534$;

$\therefore$ $\cos 34°.\,12'.\,19''\cdot6 = \cdot8270272$.

EXAMPLES—XLIV. (p. 135).

(1) $$\sin 48°.\,47' = \cdot7522233$$
$$\sin 48°.\,46' = \cdot7520316$$
$$\text{Difference for } 1' = \cdot0001917$$

$\therefore$ $\cdot0001917 : \cdot0001084 = 60''$: what we must *add* to $48°.\,46'$;

$\therefore$ we must add $34''$;

$\therefore$ the angle is $48°.\,46'.\,34''$.

(2) $\cos 2^\circ.\ 33' = \cdot 9990098$

$\cos 2^\circ.\ 34' = \cdot 9989968$

Difference for $1' = \cdot 0000130$

$\therefore\ \cdot 0000130 : \cdot 0000098 = 60'' :$ what we must add to $2^\circ\ 33'$;

$\therefore$ we must add $45''$;

$\therefore$ the angle is $2^\circ.\ 33'.\ 45''$.

(3) $\sin 43^\circ.\ 15' = \cdot 6851830$

$\sin 43^\circ.\ 14' = \cdot 6849711$

Difference for $1' = \cdot 0002119$

$\therefore\ \cdot 0002119 : \cdot 0000289 = 60'' :$ what we must add to $43^\circ.\ 14'$;

$\therefore$ we must add $8''\cdot 18$;

$\therefore$ the angle is $43^\circ.\ 14'.\ 8''\cdot 18$.

(4) $\cos 32^\circ.\ 31' = \cdot 8432351$

$\cos 32^\circ.\ 32' = \cdot 8430787$

Difference for $1' = \cdot 0001564$

$\therefore\ \cdot 0001564 : \cdot 0000351 = 60'' :$ what we must add to $32^\circ.\ 31'$;

$\therefore$ we must add $13''\cdot 46$, or, approximately, $13''\cdot 5$;

$\therefore$ the angle is $32^\circ.\ 31'.\ 13''\cdot 5$.

(5) $\sin 24^\circ.\ 12' = \cdot 4099230$

$\sin 24^\circ.\ 11' = \cdot 4096577$

Difference for $1' = \cdot 0002653$

$\therefore\ \cdot 0002653 : \cdot 0000982 = 60'' :$ what we must add to $24^\circ.\ 11'$;

$\therefore$ we must add $22''\cdot 2$;

$\therefore$ the angle is $24^\circ.\ 11'.\ 22''\cdot 2$.

(6) $\sec 82^\circ.\ 23' = 7\cdot 552169$

$\sec 82^\circ.\ 22' = 7\cdot 528249$

Difference for $1' = \ \cdot 023920$

$\therefore\ \cdot 023920 : \cdot 005084 = 60'' :$ what we must add to $82^\circ.\ 22'$;

$\therefore$ we must add $12''\cdot 8$ nearly;

$\therefore$ the angle is $82^\circ.\ 22'.\ 12''\cdot 8$.

(7) $\cos 53°.\ 7' = {\cdot}6001876$

$\cos 53°.\ 8' = \underline{{\cdot}5999549}$

Difference for $1' = {\cdot}0002327$

$\therefore\ {\cdot}0002327 : {\cdot}0001876 = 60''$: what we must add to $53°.\ 7'$;

$\therefore$ we must add $48''{\cdot}4$ nearly;

$\therefore$ the angle is $53°.\ 7'.\ 48''{\cdot}4$.

(8) $\text{cosec}\,25°.\ 3' = 2{\cdot}36179$

$\text{cosec}\,25°.\ 4' = \underline{2{\cdot}36029}$

Difference for $1' = \ {\cdot}00150$

$\therefore\ {\cdot}00150 : {\cdot}00068 = 60''$: what we must add to $25°.\ 3'$;

$\therefore$ we must add $27''{\cdot}2$;

$\therefore$ the angle is $25°.\ 3'.\ 27''{\cdot}2$.

(9) $\sin 73°.\ 45' = {\cdot}9600499$

$\sin 73°.\ 44' = \underline{{\cdot}9599684}$

Difference for $1' = {\cdot}0000815$

$\therefore\ {\cdot}0000815 : {\cdot}0000316 = 60''$: what we must add to $73°.\ 44'$;

$\therefore$ we must add $23''{\cdot}2$;

$\therefore$ the angle is $73°.44'.\ 23''{\cdot}2$.

(10) $\tan 77°.\ 20' = 4{\cdot}44942$

$\tan 77°.\ 19' = \underline{4{\cdot}44338}$

Difference for $1' = \ {\cdot}00604$

$\therefore\ {\cdot}00604 : 00106 = 60''$: what we must add to $77°.\ 19'$;

$\therefore$ we must add $10''{\cdot}5$;

$\therefore$ the angle is $77°.\ 19'.\ 10''{\cdot}5$.

EXAMPLES—XLV. (p. 138).

(1) $L \sin 55°.\ 34' = 9{\cdot}9163406$

$L \sin 55°.\ 33' = \underline{9{\cdot}9162539}$

Difference for $1' = \ {\cdot}0000867$

$\therefore\ 60'' : 54'' = {\cdot}0000867$: what we have to add;

$\therefore$ we must add ${\cdot}0000780$;

$\therefore\ L \sin 55°.\ 33'.\ 54'' = 9{\cdot}9163319$.

(2) $L\sin 29°.\,26' = 9{\cdot}6914445$

$L\sin 29°.\,25' = 9{\cdot}6912205$

Difference for $1' = {\cdot}0002240$

$\therefore\ 60'' : 2'' = {\cdot}0002240$: what we have to add;

$\therefore$ we must add ${\cdot}0000075$;

$\therefore\ L\sin 29°.\,25'.\,2'' = 9{\cdot}6912280.$

(3) $L\cos 37°.\,28' = 9{\cdot}8996604$

$L\cos 37°.\,29' = 9{\cdot}8995636$

Difference for $1' = {\cdot}0000968$

$\therefore\ 60'' : 36'' = {\cdot}0000968$: what we have to subtract;

$\therefore$ we must subtract ${\cdot}0000581$;

$\therefore\ L\cos 37°.\,28'.\,36'' = 9{\cdot}8996023.$

(4) $L\sin 54°.\,14' = 9.9092371$

$L\sin 54°.\,13' = 9{\cdot}9091461$

Difference for $1' = {\cdot}0000910$

$\therefore\ 60'' : 19'' = {\cdot}0000910$: what we have to add;

$\therefore$ we must add ${\cdot}0000288$;

$\therefore\ L\sin 54°.\,13'.\,19'' = 9{\cdot}9091749.$

(5) $L\tan 27°.\,43' = 9{\cdot}7204759$

$L\tan 27°.\,42' = 9{\cdot}7201690$

Difference for $1' = {\cdot}0003069$

$\therefore\ 60'' : 34'' = {\cdot}0003069$: what we have to add;

$\therefore$ we must add ${\cdot}0001739$;

$\therefore\ L\tan 27''.\,42'.\,34'' = 9{\cdot}7203429.$

(6) $L\tan 5°.\,14' = 8{\cdot}9618659$

$L\tan 5°.\,13' = 8{\cdot}9604728$

Difference for $1' = {\cdot}0013931$

$\therefore\ 60'' : 23'' = {\cdot}0013931$: what we have to add;

$\therefore$ we must add ${\cdot}0005340$;

$\therefore\ L\tan 5°.\,13'.\,23'' = 8{\cdot}9610068.$

(7)
$$L\cot 3^\circ.\,37' = 11{\cdot}1992368$$
$$L\cot 3^\circ.\,38' = \underline{11{\cdot}1972347}$$
$$\text{Difference for } 1' = \quad {\cdot}0020021$$
$\therefore$ $60'' : 50'' = {\cdot}0020021$: what we have to subtract;
$\therefore$ we must subtract ·0016684;
$\therefore$ $L\cot 3^\circ.\,37'.\,50'' = 11{\cdot}1975684.$

(8)
$$L\sin 39^\circ.\,26' = 9{\cdot}8028968$$
$$L\sin 39^\circ.\,25' = \underline{9{\cdot}8027431}$$
$$\text{Difference for } 1' = \quad {\cdot}0001537$$
$\therefore$ $60'' : 10'' = {\cdot}0001537$: what we have to add;
$\therefore$ we must add ·0000256;
$\therefore$ $L\sin 39^\circ.\,25'.\,10'' = 9{\cdot}8027687.$

(9)
$$L\sin 70^\circ.\,35' = 9{\cdot}9745697$$
$$L\sin 70^\circ.\,34' = \underline{9{\cdot}9745252}$$
$$\text{Difference for } 1' = \quad {\cdot}0000445$$
$\therefore$ $60'' : 17'' = {\cdot}0000445$: what we must add;
$\therefore$ we must add ·0000126;
$\therefore$ $L\sin 70^\circ.\,34'.\,17'' = 9{\cdot}9745378.$

(10)
$$L\cos 88^\circ.\,54' = 8{\cdot}2832434$$
$$L\cos 88^\circ.\,55' = \underline{8{\cdot}2766136}$$
$$\text{Difference for } 1' = \quad {\cdot}0066298$$
$\therefore$ $60'' : 16'' = {\cdot}0066298$: what we must subtract;
$\therefore$ we must subtract ·0017679;
$\therefore$ $L\cos 88^\circ.\,54'.\,16'' = 8{\cdot}2814755.$

Examples—XLVI. (p. 140).

(1)
$$L\sin 14^\circ.\,25' = 9{\cdot}3961499$$
$$L\sin 14^\circ.\,24' = \underline{9{\cdot}3956581}$$
$$\text{Difference for } 1' = \quad {\cdot}0004918$$
$\therefore$ ${\cdot}0004918 : {\cdot}0002868 = 60^\circ$: what we have to add;
$\therefore$ we must add $35''$ nearly;
$\therefore$ the angle is $14^\circ.\,24'.\,35''$.

(2) $L \sin 54°. 14' = 9{\cdot}9092371$

$L \sin 54°. 13' = 9{\cdot}9091461$

Difference for $1' = {\cdot}0000910$

$\therefore$ 0000910 : ·0000299 = 60″ : what we have to add;

$\therefore$ we must add 19″;

$\therefore$ the angle is 54°. 13′. 19″.

(3) $L \sin 71°. 41' = 9{\cdot}9774191$

$L \sin 71°. 40' = 9{\cdot}9773772$

Difference for $1' = {\cdot}0000419$

$\therefore$ ·0000419 : ·0000125 = 60″ : what we must add;

$\therefore$ we must add 18″ nearly;

$\therefore$ the angle is 71°. 40′. 18″.

(4) $L \cos 29°. 25' = 9{\cdot}9400535$

$L \cos 29°. 26' = 9{\cdot}9399823$

Difference for $1' = {\cdot}0000712$

$\therefore$ ·0000712 : ·0000023 = 60″ : what we must add;

$\therefore$ we must add 2″ nearly;

$\therefore$ the angle is 29°. 25′. 2″.

(6) $L \tan 30°. 51' = 9{\cdot}7761947$

$L \tan 30°. 50' = 9{\cdot}7759077$

Difference for $1' = {\cdot}0002870$

$\therefore$ ·0002870 : ·0001320 = 60′ : what we must add;

$\therefore$ we must add 27″·6 nearly;

$\therefore$ the angle is 30°. 50′. 27″·6.

(6) $L \cot 86°. 32' = 8{\cdot}7823199$

$L \cot 86°. 33' = 8{\cdot}7802218$

Difference for $1' = {\cdot}0020981$

$\therefore$ ·0020981 : ·0008556 = 60″ : what we must add;

$\therefore$ we must add 24″·5 nearly;

$\therefore$ the angle is 86°. 32′. 24″·5.

(7) $L\sin 24°.\,9' = 9{\cdot}6118580$

$L\sin 24°.\,8' = 9{\cdot}6115762$

Difference for $1' = \;{\cdot}0002818$

$\therefore\ {\cdot}0002818 : {\cdot}0002114 = 60''$: what we must add;

$\therefore$ we must add $45''$;

$\therefore$ the angle is $24°.\,8'.\,45''$.

(8) $L\tan 11°.\,40' = 9{\cdot}3148851$

$L\tan 11°.\,39' = 9{\cdot}3142468$

Difference for $1' = \;{\cdot}0006383$

$\therefore\ {\cdot}0006383 : {\cdot}0005543 = 60''$: what we must add;

$\therefore$ we must add $52''$;

$\therefore$ the angle is $11°.\,39'.\,52''$.

(9) $L\operatorname{cosec} 46°.\,23' = 10{\cdot}1402787$

$L\operatorname{cosec} 46°.\,24' = 10{\cdot}1401584$

Difference for $1' = \;{\cdot}0001203$

$\therefore\ {\cdot}0001203 : {\cdot}0000220 = 60''$: what we must add;

$\therefore$ we must add $11''$ nearly;

$\therefore$ the angle is $46°.\,23'.\,11''$.

(10) $L\sec 29°.\,55' = 10{\cdot}0621053$

$L\sec 29°.\,54' = 10{\cdot}0620326$

Difference for $1' = \;{\cdot}0000727$

$\therefore\ {\cdot}0000727 : {\cdot}0000359 = 60''$: what we must add;

$\therefore$ we must add $29''{\cdot}6$ nearly;

$\therefore$ the angle is $29°.\,54'.\,29''{\cdot}6$.

Examples—XLVII. (p. 149).

(1) $\sin(A+B) = \sin(180° - C) = \sin C$.

(2) $\cos(A+B) = \cos(180° - C) = -\cos C$.

(3) $\sin\dfrac{A+B}{2} = \sin\left(90° - \dfrac{C}{2}\right) = \cos\dfrac{C}{2}$.

(4) $\cos\frac{A+B}{2}=\cos\left(90^\circ-\frac{C}{2}\right)=\sin\frac{C}{2}.$

(5) $\tan\frac{A+B}{2}=\tan\left(90^\circ-\frac{C}{2}\right)=\cot\frac{C}{2}.$

(6) $\cot\frac{A+B}{2}=\cot\left(90^\circ-\frac{C}{2}\right)=\tan\frac{C}{2}.$

EXAMPLES—XLVIII. (p. 150).

1. (1) $\sin 2A+\sin 2B+\sin 2C=2\sin(A+B).\cos(A-B)+\sin 2C$

$$=2\sin C.\cos(A-B)+2\sin C.\cos C$$
$$=2\sin C.\{\cos(A-B)+\cos C\}$$
$$=2\sin C\{\cos(A-B)-\cos(A+B)\}$$
$$=2\sin C.(2\sin A.\sin B)$$
$$=4\sin A.\sin B.\sin C.$$

(2) $\sin(-A+B+C)+\sin(A-B+C)+\sin(A+B-C)$

$$=2\sin C.\cos(A-B)+\sin(A+B).\cos C-\cos(A+B).\sin C$$
$$=2\sin C.\cos(A-B)+\sin C.\cos C+\cos C.\sin C$$
$$=2\sin C.\{\cos(A-B)+\cos C\}$$
$$=2\sin C.\{\cos(A-B)-\cos(A+B)\}$$
$$=4\sin A.\sin B.\sin C.$$

(3)
$$\frac{\cot\frac{A}{2}+\cot\frac{C}{2}}{\cot\frac{B}{2}+\cot\frac{C}{2}}=\frac{\dfrac{\cos\frac{A}{2}\cdot\sin\frac{C}{2}+\sin\frac{A}{2}\cdot\cos\frac{C}{2}}{\sin\frac{A}{2}\cdot\sin\frac{C}{2}}}{\dfrac{\cos\frac{B}{2}\cdot\sin\frac{C}{2}+\sin\frac{B}{2}\cdot\cos\frac{C}{2}}{\sin\frac{B}{2}\cdot\sin\frac{C}{2}}}=\frac{\sin\left(\frac{A}{2}+\frac{C}{2}\right)\cdot\sin\frac{B}{2}}{\sin\left(\frac{B}{2}+\frac{C}{2}\right)\cdot\sin\frac{A}{2}}$$

$$=\frac{\cos\frac{B}{2}\cdot\sin\frac{B}{2}}{\cos\frac{A}{2}\cdot\sin\frac{A}{2}}=\frac{2\cos\frac{B}{2}\cdot\sin\frac{B}{2}}{2\cos\frac{A}{2}\cdot\sin\frac{A}{2}}=\frac{\sin B}{\sin A}.$$

(4) $\tan(A+B+C)=\tan 180°=0$;

$$\therefore \frac{\tan A+\tan B+\tan C-\tan A\,.\,\tan B\,.\,\tan C}{1-\tan A\,.\,\tan B-\tan B\,.\,\tan C-\tan C\,.\,\tan A}=0;$$

$$\therefore \tan A+\tan B+\tan C-\tan A\,.\,\tan B\,.\,\tan C=0;$$

$$\therefore \tan A+\tan B+\tan C=\tan A\,.\,\tan B\,.\,\tan C.$$

(5) As in Example (4),

$$\tan A+\tan B+\tan C=\tan A\,.\,\tan B\,.\,\tan C,$$

and dividing both sides by $\tan A\,.\,\tan B\,.\,\tan C$,

$$\cot B\,.\,\cot C+\cot A\,.\,\cot C+\cot A\,.\,\cot B=1.$$

(6) $$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{\cos\frac{A}{2}\cdot\sin\frac{B}{2}+\sin\frac{A}{2}\cdot\cos\frac{B}{2}}{\sin\frac{A}{2}\cdot\sin\frac{B}{2}}+\frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}.$$

$$=\frac{\sin\left(\frac{A}{2}+\frac{B}{2}\right)}{\sin\frac{A}{2}\cdot\sin\frac{B}{2}}+\frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}$$

$$=\cos\frac{C}{2}\left\{\frac{1}{\sin\frac{A}{2}\cdot\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}\right\}$$

$$=\cos\frac{C}{2}\left\{\frac{\sin\frac{C}{2}+\sin\frac{A}{2}\cdot\sin\frac{B}{2}}{\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}}\right\}$$

$$=\cos\frac{C}{2}\left\{\frac{\cos\left(\frac{A}{2}+\frac{B}{2}\right)+\sin\frac{A}{2}\cdot\sin\frac{B}{2}}{\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}}\right\}$$

$$=\frac{\cos\frac{C}{2}\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}}{\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}}=\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}.$$

(7)

$$1+\cos 2A+\cos 2B+\cos 2C=1+(2\cos^2 A-1)+2\cos(B+C).\cos(B-C)$$
$$=2\cos^2 A-2\cos A.\cos(B-C)$$
$$=-2\cos A.\{\cos(B+C)+\cos(B-C)\}$$
$$=-2\cos A.2\cos B.\cos C=-4\cos A.\cos B.\cos C.$$

(8) $$\cos A+\cos B+\cos C=2\cos\frac{A+B}{2}\cdot\cos\frac{A-B}{2}+1-2\sin^2\frac{C}{2}$$
$$=2\sin\frac{C}{2}\cdot\cos\frac{A-B}{2}-2\sin^2\frac{C}{2}+1$$
$$=2\sin\frac{C}{2}\left\{\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right\}+1$$
$$=2\sin\frac{C}{2}\cdot 2\sin\frac{A}{2}\cdot\sin\frac{B}{2}+1=4\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}+1.$$

(9) $$-\sin 2A+\sin 2B+\sin 2C=2\sin(B+C).\cos(B-C)-2\sin A.\cos A$$
$$=2\sin A.\{\cos(B-C)-\cos A\}$$
$$=2\sin A.\{\cos(B-C)+\cos(B+C)\}$$
$$=4\sin A.\cos B.\cos C.$$

(10) $$\sin A+\sin B-\sin C=2\sin\frac{A+B}{2}\cdot\cos\frac{A-B}{2}-2\sin\frac{C}{2}\cdot\cos\frac{C}{2}$$
$$=2\cos\frac{C}{2}\cdot\left\{\cos\frac{A-B}{2}-\sin\frac{C}{2}\right\}$$
$$=2\cos\frac{C}{2}\cdot\left\{\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right\}$$
$$=2\cos\frac{C}{2}\cdot 2\sin\frac{A}{2}\cdot\sin\frac{B}{2}$$
$$=4\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\cos\frac{C}{2}.$$

(11) $$\sin 2A+\sin 2B-\sin 2C=2\sin(A+B).\cos(A-B)-2\sin C.\cos C$$
$$=2\sin C.\{\cos(A-B)-\cos C\}$$
$$=2\sin C.\{\cos(A-B)+\cos(A+B)\}$$
$$=4\sin C.\cos A.\cos B.$$

(12) $\cos A+\cos B-\cos C=2\cos\frac{A+B}{2}\cdot\cos\frac{A-B}{2}-\left(1-2\sin^2\frac{C}{2}\right)$

$$=2\sin\frac{C}{2}\cdot\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cdot\cos\frac{A+B}{2}-1$$

$$=2\sin\frac{C}{2}\cdot\left\{\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right\}-1$$

$$=4\sin\frac{C}{2}\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}-1.$$

(13) $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=\frac{1}{2}\left\{\cos A+1+\cos B+1+\cos C+1\right\}$

$$=\frac{1}{2}\cdot\left\{\cos A+\cos B+\cos C+3\right\}$$

$$=\frac{1}{2}\cdot\left\{4\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}+1+3\right\}, \text{ as in Ex. 8.}$$

$$=2+2\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}.$$

(14) $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=\frac{1}{2}\cdot\left\{1-\cos A+1-\cos B+1-\cos C\right\}$

$$=\frac{1}{2}\cdot\left\{3-4\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}-1\right\}, \text{ as in Ex. 8.}$$

$$=1-2\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}.$$

2.

(1) $\frac{b+c}{a}=\cot A+\operatorname{cosec}A=\frac{\cos A+1}{\sin A}=\frac{2\cos^2\frac{A}{2}}{2\sin\frac{A}{2}\cdot\cos\frac{A}{2}}=\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}=\cot\frac{A}{2}.$

(2) $2\operatorname{cosec}2A\,.\cot B=\frac{2}{\sin 2A}\cdot\frac{\cos B}{\sin B}=\frac{2\cos B}{2\sin A\,.\cos A\,.\sin B}$

$$=\frac{\cos B}{\cos B\,.\sin B\,.\sin B}=\frac{1}{\sin^2 B}=\frac{c^2}{b^2}$$

(3) $$2\sin^2\frac{B}{2}=1-\cos B=1-\frac{a}{c}=\frac{c-a}{c};$$

$$\therefore \sin\frac{B}{2}=\sqrt{\left(\frac{c-a}{2c}\right)}.$$

(4) $$2\cos^2\frac{B}{2}=1+\cos B=1+\frac{a}{c}=\frac{a+c}{c};$$

$$\therefore \cos\frac{B}{2}=\sqrt{\left(\frac{a+c}{2c}\right)}.$$

(5) $$\frac{\cos 2B-\cos 2A}{\sin 2A}=\frac{\cos^2 B-\sin^2 B-\cos^2 A+\sin^2 A}{2\sin A\,.\cos A}$$

$$=\frac{\sin^2 A-\sin^2 B-\sin^2 B+\sin^2 A}{2\sin A\,.\cos A}=\frac{2\sin^2 A-2\sin^2 B}{2\sin A\,.\cos A}$$

$$=\frac{\sin A}{\cos A}-\frac{\sin B}{\cos B}=\tan A-\tan B.$$

(6) $$\tan 2A-\sec 2B=\frac{2\tan A}{1-\tan^2 A}-\frac{1}{\cos^2 B-\sin^2 B}$$

$$=\frac{2ab}{b^2-a^2}-\frac{c^2}{a^2-b^2}=\frac{2ab+c^2}{b^2-a^2}$$

$$=\frac{2ab+a^2+b^2}{b^2-a^2}=\frac{b+a}{b-a}.$$

(7) $(\sin A-\sin B)^2+(\cos A+\cos B)^2$

$$=\sin^2 A-2\sin A\,.\sin B+\sin^2 B+\cos^2 A+2\cos A\,.\cos B+\cos^2 B$$

$$=2+2(\cos A\,.\cos B-\sin A\,.\sin B)$$

$$=2+2\cos(A+B)=2-2\cos C=4\sin^2\frac{C}{2}.$$

(8) $$\sec 2A=\frac{1}{\cos^2 A-\sin^2 A}=\frac{1}{\frac{b^2-a^2}{c^2}}=\frac{c^2}{b^2-a^2}.$$

(9) $a^3 . \cos A + b^3 . \cos B = a^3 . \frac{b}{c} + b^3 . \frac{a}{c} = \frac{ab(a^2+b^2)}{c} = \frac{abc^2}{c} = abc.$

(10)

$$\cot(B-A) + \cot 2\left(A + \frac{C}{2}\right) = \frac{\cos B . \cos A + \sin B . \sin A}{\sin B . \cos A - \cos B . \sin A} + \cot(2A + 90^\circ)$$

$$= \frac{\sin A . \sin B + \sin B . \sin A}{\sin B . \sin B - \sin A . \sin A} - \tan 2A$$

$$= \frac{2ab}{b^2 - a^2} - \frac{2 \tan A}{1 - \tan^2 A} = \frac{2ab}{b^2 - a^2} - \frac{2ab}{b^2 - a^2} = 0.$$

3. (1) $\frac{\sin A - \sin B}{a - b} = \frac{\frac{a \sin C}{c} - \frac{b \sin C}{c}}{a - b} = \frac{(a-b)\sin C}{(a-b)c} = \frac{\sin C}{c}.$

(2) $\frac{\sin(A-B)}{\sin C} = \frac{\sin(A-B) . \sin(A+B)}{\sin C . \sin C} = \frac{\sin^2 A - \sin^2 B}{\sin^2 C} = \frac{a^2 - b^2}{c^2}.$

(3) $\frac{a . \sin C}{b - a \cos C} = \frac{a \sin C}{a \cos C + c \cos A - a \cos C} = \frac{a . \sin C}{c . \cos A} = \frac{c . \sin A}{c . \cos A} = \tan A.$

(4) $\frac{c}{a} \cdot \operatorname{cosec} B - \cot B = \frac{c}{a . \sin B} - \frac{\cos B}{\sin B} = \frac{c - a . \cos B}{a . \sin B}$

$$= \frac{b \cos A + a \cos B - a \cos B}{a \sin B} = \frac{b \cos A}{b \sin A} = \cot A.$$

(5) $a + b + c = (b \cos C + c \cos B) + (a \cos C + c \cos A) + (a \cos B + b \cos A)$

$$= (a+b)\cos C + (a+c)\cos B + (b+c)\cos A.$$

(6) $\frac{a+b}{c} = \frac{\sin A + \sin B}{\sin C} = \frac{2 \sin\frac{A+B}{2} \cdot \cos\frac{A-B}{2}}{2 \sin\frac{C}{2} \cdot \cos\frac{C}{2}} = \frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}};$

$$\therefore (a+b) . \sin\frac{C}{2} = c . \cos\frac{A-B}{2}.$$

(7) $\dfrac{a-b}{c}=\dfrac{\sin A-\sin B}{\sin C}=\dfrac{2\cos\frac{A+B}{2}\cdot\sin\frac{A-B}{2}}{2\sin\frac{C}{2}\cdot\cos\frac{C}{2}}=\dfrac{\sin\frac{A-B}{2}}{\cos\frac{C}{2}}$;

$$\therefore\ (a-b)\cos\frac{C}{2}=c\,.\,\sin\frac{A-B}{2}.$$

(8) $\dfrac{\tan B}{\tan C}=\dfrac{\sin B\,.\,\cos C}{\sin C\,.\,\cos B}=\dfrac{b\,.\left(\frac{a^2+b^2-c^2}{2ab}\right)}{c\,.\left(\frac{a^2+c^2-b^2}{2ac}\right)}=\dfrac{a^2+b^2-c^2}{a^2-b^2+c^2}.$

(9) $c=a\cos B+b\cos A=a\cos B+\dfrac{a\sin B}{\sin A}\cdot\cos A=a(\cos B+\sin B.\cot A).$

(10) $2(ab\,.\,\cos C+ac\,.\,\cos B+bc\,.\,\cos A)$
$=(a^2+b^2-c^2)+(a^2+c^2-b^2)+(b^2+c^2-a^2)=a^2+b^2+c^2.$

(11) $\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\,.\,\cos B\,.\,\cos C$

$=\dfrac{1}{2}\left\{1+\cos 2A+1+\cos 2B+1+\cos 2C+4\cos A\,.\,\cos B\,.\,\cos C\right\}$

$=\dfrac{1}{2}\left\{3+(-1-4\cos A\,.\,\cos B\,.\,\cos C)+4\cos A\,.\,\cos B\,.\,\cos C\right\}$,

by Example XLVIII. 1. (7).

$=\dfrac{1}{2}\times 2=1.$

(12)

$\dfrac{a-b}{c}\cdot 2\cos^2\dfrac{C}{2}=\dfrac{\sin A-\sin B}{\sin C}\cdot 2\cos^2\dfrac{C}{2}=\dfrac{2\cos\frac{A+B}{2}\cdot\sin\frac{A-B}{2}}{\sin\frac{C}{2}}\cdot\cos\dfrac{C}{2}$

$$=2\sin\frac{A-B}{2}\cdot\sin\frac{A+B}{2}=\cos B-\cos A.$$

(13)

$\dfrac{a+b}{c}\cdot 2\sin^2\dfrac{C}{2}=\dfrac{\sin A+\sin B}{\sin C}\cdot 2\sin^2\dfrac{C}{2}=\dfrac{2\sin\frac{A+B}{2}\cdot\cos\frac{A-B}{2}}{\cos\frac{C}{2}}\cdot\sin\dfrac{C}{2}$

$$=2\cos\frac{A-B}{2}\cdot\cos\frac{A+B}{2}=\cos A+\cos B.$$

(14) $a^2 . \sin A + ab . \sin B + ac . \sin C = a^2 \sin A + b . b \sin A + c . c \sin A$
$$= (a^2 + b^2 + c^2)\sin A.$$

(15) By Art. 184, page 149,
$$\cot\frac{A}{2} = \sqrt{\frac{s . (s-a)}{(s-b)(s-c)}} \text{ and } \cot\frac{B}{2} = \sqrt{\frac{s . (s-b)}{(s-a)(s-c)}};$$
$$\therefore \cot\frac{A}{2} : \cot\frac{B}{2} = s - a : s - b$$
$$= b + c - a : a + c - b.$$

(16)
$$\cot\frac{A}{2} \cdot \cot\frac{B}{2} = \sqrt{\frac{s . (s-a)}{(s-b)(s-c)}} \cdot \sqrt{\frac{s . (s-b)}{(s-a)(s-c)}} = \frac{s}{s-c} = \frac{a+b+c}{a+b-c}.$$

(17) $a \sin(B - C) + b \sin(C - A) + c . \sin(A - B)$
$= a (\sin B . \cos C - \cos B . \sin C) + b (\sin C . \cos A - \cos C . \sin A)$
$+ c (\sin A . \cos B - \cos A . \sin B)$
$= \cos C(a \sin B - b \sin A) + \cos B(c \sin A - a \sin C)$
$+ \cos A(b \sin C - c \sin B)$
$= 0 + 0 + 0 = 0.$

4. If the sides are in arithmetical progression, so also are the sines of the angles :
$$\therefore \sin A + \sin C = 2 \sin B,$$
$$\text{or } \sin A + \sin(A + B) = 2 \sin B,$$
$$\text{or } 2 \sin\left(A + \frac{B}{2}\right) \cos\frac{B}{2} = 4 \sin\frac{B}{2} \cdot \cos\frac{B}{2};$$
$$\therefore \sin\left(A + \frac{B}{2}\right) = 2 \sin\frac{B}{2}.$$

5. $(b + c) . AD = b . AD + c . AD$
$= b . b \sin C + c . c \sin B$
$= b^2 \sin C + c^2 \sin B.$

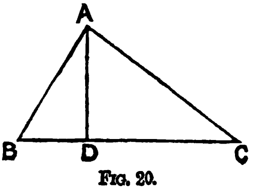

FIG. 20.

6. Let $AB=4, AC=9, BC=12$, and let AD be the line bisecting $\angle BAC$.

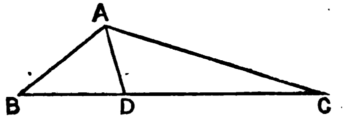

FIG. 21.

Then, by EUCLID VI. *B*,

$$BD.DC+DA^2=BA.AC$$

$$AD\cdot\frac{\sin\frac{A}{2}}{\sin B}\times AD\cdot\frac{\sin\frac{A}{2}}{\sin C}+DA^2=36$$

$$AD^2\left(\frac{\sin^2\frac{A}{2}}{\sin B\cdot\sin C}+1\right)=36$$

$$AD^2\left\{\frac{\frac{(s-b)(s-c)}{bc}}{\frac{4}{a^2bc}\cdot s.(s-a).(s-b).(s-c)}+1\right\}=36$$

$$AD^2\left\{\frac{a^2}{4.s.(s-a)}+1\right\}=36$$

$$AD^2\times\frac{169}{25}=36,\ \text{or,}\ AD=\frac{6\times5}{13}=2\frac{4}{13}.$$

7. $$\text{If }\sin A=2\cos B.\sin C$$

$$\sin(B+C)=2\cos B.\sin C$$

$$\sin B.\cos C+\cos B.\sin C=2\cos B.\sin C$$

$$\sin B.\cos C-\cos B\sin C=0$$

$$\sin(B-C)=0,\ \text{and}\ \therefore\ B=C.$$

8. If $\cos A \,.\, \cos B \,.\, \sin C = \dfrac{\dfrac{\sin A + \sin B}{\cos A + \cos B}}{\cos A \,.\, \cos B}$

$$\sin C = \frac{\sin A + \sin B}{\cos A + \cos B} = \frac{2 \,.\, \sin\frac{A+B}{2} \cdot \cos\frac{A-B}{2}}{2 \,.\, \cos\frac{A+B}{2} \cdot \cos\frac{A-B}{2}};$$

$$\therefore\ 2 \sin\frac{C}{2} \cdot \cos\frac{C}{2} = \frac{\cos\frac{C}{2}}{\sin\frac{C}{2}};$$

$$\therefore\ \sin^2\frac{C}{2} = \frac{1}{2}, \text{ or, } \sin\frac{C}{2} = \frac{1}{\sqrt{2}};$$

$$\therefore\ \frac{C}{2} = 45°, \text{ and } \therefore\ C = 90°.$$

9. $$\text{If } \sin^2 A = \sin^2 B + \sin^2 C$$

$$\sin^2 A = \frac{b^2}{a^2} \cdot \sin^2 A + \frac{c^2}{a^2} \cdot \sin^2 A;$$

$$\therefore\ a^2 = b^2 + c^2, \text{ and } \therefore\ A = 90°.$$

10. $$\text{If } \frac{\sin A}{\sin C} = \frac{\sin C}{\sin B}, \text{ then } \frac{a}{c} = \frac{c}{b}, \text{ or, } ab = c^2.$$

$$\text{Then } \frac{a^3 + b^3 + c^3}{a + b + c} = ab$$

$$a^3 + b^3 + c^3 = ab(a+b) + abc$$

$$= ab(a+b) + c^3;$$

$$\therefore\ a^3 + b^3 = ab(a+b);$$

$$\therefore\ a^2 - ab + b^2 = ab, \text{ or, } (a-b)^2 = 0, \text{ or, } a = b.$$

Hence a, b, c are all equal.

11. $$c^2 = a^2 + b^2 - 2ab \,.\, \cos C$$

$$= a^2 + b^2 - 2ab \times \left(-\frac{1}{2}\right)$$

$$= a^2 + b^2 + ab.$$

12. $\dfrac{\sin A}{\sin B}=\dfrac{a}{b}$;

$\therefore \dfrac{\sin A+\sin B}{\sin A-\sin B}=\dfrac{a+b}{a-b}$;

$\therefore \dfrac{\sin(B+C)+\sin B}{\sin(B+C)-\sin B}=\dfrac{a+b}{a-b}$;

$\therefore \dfrac{\sin\left(B+\dfrac{C}{2}\right)\cdot\cos\dfrac{C}{2}}{\cos\left(B+\dfrac{C}{2}\right)\cdot\sin\dfrac{C}{2}}=\dfrac{a+b}{a-b}$.

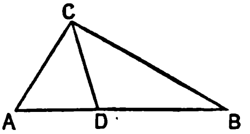

FIG. 22.

Now $\angle ADC=B+\dfrac{C}{2}$, by EUCLID I. 32

$$\therefore \tan ADC.\cot\frac{C}{2}=\frac{a+b}{a-b};$$

$$\therefore \tan ADC=\frac{a+b}{a-b}\cdot\tan\frac{C}{2}.$$

(13) Draw CE perpendicular to AB.

Then by EUCLID II. XII. and XIII.

$CB^2=CD^2+DB^2+2DB, DE,$

$CA^2=CD^2+DA^2-2AD, DE,$

and $DB=AD$.

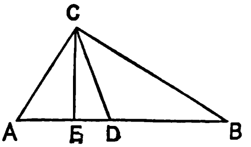

FIG. 23.

$$\therefore\ CB^2+CA^2=2\ CD^2+DB^2+DA^2;$$

$$\therefore\ a^2+b^2=2\ CD^2+\frac{c^2}{4}+\frac{c^2}{4};$$

$$\therefore\ CD^2=\frac{a^2}{2}+\frac{b^2}{2}-\frac{c^2}{4}.$$

EXAMPLES—XLIX. (p. 157).

(1) $$a=\sqrt{c^2-b^2}=\sqrt{16}=4$$

$$\sin A=\frac{a}{c}=\frac{4}{5}=\cdot 8.$$

Hence, as in Art. 168, we find $A=53°.\,7'.\,48''\cdot 4$;

and $\therefore\ B=36°.\,52'.\,11''\cdot 6$.

(2) $$a=\sqrt{c^2-b^2}=\sqrt{64}=8,$$

$$\sin A=\frac{a}{c}=\frac{8}{17}=\cdot 4705882.$$

Hence $A=28°.\ 4'.\ 20''\cdot 9$, and $B=61°.\ 55'.\ 39''\cdot 1$.

(3) $$a=\sqrt{c^2-b^2}=\sqrt{400}=20,$$

$$\sin A=\frac{a}{c}=\frac{20}{29}=\cdot 6896552.$$

Hence $A=43°.\ 36'.\ 10''\cdot 1$, and $B=46°.\ 23'.\ 49''\cdot 9$.

(4) $$a=\sqrt{c^2-b^2}=\sqrt{576}=24,$$

$$\cos A=\frac{b}{c}=\frac{7}{25}=\cdot 28.$$

Hence $A=73°.\ 44'.\ 23''.\ 3$, and $B=16°.\ 15'.\ 36''\cdot 7$.

(5) $$a=\sqrt{c^2-b^2}=\sqrt{3136}=56,$$

$$\cos A=\frac{b}{c}=\frac{33}{65}=\cdot 5076923.$$

$\therefore\ A=59°.\ 29'.\ 23''\cdot 2$, and $B=30°.\ 30'.\ 36''\cdot 8$.

(6) $a=c\,.\sin A=13\times\cdot 9230770=12$ very nearly,
$b=\sqrt{c^2-a^2}=\sqrt{25}=5,$
$B=22°.\ 37'.\ 11''\cdot 5$.

(7) $a=c\,.\sin A=41\times\cdot 9756098=40$ very nearly,
$b=\sqrt{c^2-a^2}=\sqrt{81}=9,$
$B=12°.\ 40'.\ 49''.\ 4$.

(8) $a=c\,.\cos B=73\times\cdot 6575341=48$ very nearly,
$b=\sqrt{c^2-a^2}=\sqrt{3025}=55,$
$A=41°.\ 6'.\ 43''\cdot 5$.

(9) $a=c\,.\cos B=89\times\cdot 4382021=39$ very nearly,
$b=\sqrt{c^2-a^2}=\sqrt{6400}=80,$
$A=25°.\ 59'.\ 21''\cdot 2$.

(10) $b = a \div \tan A = 40 \div 4{\cdot}4444442 = 9$ very nearly,

$c = \sqrt{a^2 + b^2} = \sqrt{1681} = 41,$

$B = 12°.\ 40'.\ 49''{\cdot}4.$

Examples—L. (p. 159).

(1) $$b = \sqrt{c^2 - a^2} = \sqrt{289 \times 81} = 17 \times 9 = 153,$$

$$\sin A = \frac{a}{c},$$

$$L \sin A = 10 + 2{\cdot}0170333 - 2{\cdot}2671717 = 9{\cdot}7498616;$$

$$\therefore\ A = 34°.\ 12'.\ 19''{\cdot}6, \text{ and } B = 55°.\ 47'.\ 40''{\cdot}4.$$

(2) $$b = \sqrt{c^2 - a^2} = \sqrt{729 \times 121} = 27 \times 11 = 297,$$

$$\sin A = \frac{a}{c};$$

$$\therefore\ L \sin A = 10 + 2{\cdot}4828736 - 2{\cdot}6283889 = 9{\cdot}8544847;$$

$$\therefore\ A = 45°.\ 40'.\ 2''{\cdot}3, \text{ and } B = 44°.\ 19'.\ 57''{\cdot}7.$$

(3) $$b = \sqrt{c^2 - a^2} = \sqrt{1681 \times 1} = 41,$$

$$\sin A = \frac{a}{c};$$

$$\therefore\ L \sin A = 10 + 2{\cdot}9242793 - 2{\cdot}9247960 = 9{\cdot}9994833;$$

$$\therefore\ A = 87°.\ 12'.\ 20''{\cdot}3, \text{ and } B = 2°.\ 47'.\ 39''{\cdot}7.$$

(4) $$b = \sqrt{c^2 - a^2} = \sqrt{961 \times 289} = 31 \times 17 = 527,$$

$$\sin A = \frac{a}{c};$$

$$\therefore\ L \sin A = 10 + 2{\cdot}5263393 - 2{\cdot}7958800 = 9{\cdot}7304593;$$

$$\therefore\ A = 32°.\ 31'.\ 13''{\cdot}5, \text{ and } B = 57°.\ 28'.\ 46''{\cdot}5.$$

(5) $$b = \sqrt{c^2 - a^2} = \sqrt{2209 \times 9} = 47 \times 3 = 141,$$

$$\sin A = \frac{a}{c};$$

$$\therefore\ L \sin A = 10 + 3{\cdot}0413927 - 3{\cdot}0449315 = 9{\cdot}9964612;$$

$$\therefore\ A = 82°.\ 41'.\ 44'', \text{ and } B = 7°.\ 18'.\ 16''.$$

(6) $$a = \sqrt{c^2 - b^2} = \sqrt{968 \times 578};$$

$$\therefore \log a = \frac{1}{2}\{\log 968 + \log 578\} = 2{\cdot}8739016;$$

$$\therefore a = 748, \text{ and } \cos A = \frac{b}{c};$$

$$\therefore L\cos A = 10 + 2{\cdot}2900346 - 2{\cdot}8881795 = 9{\cdot}4018551.$$

$$\therefore A = 75^\circ.\ 23'.\ 18''{\cdot}5, \text{ and } B = 14^\circ.\ 36'.\ 41''{\cdot}5.$$

(7) $$a = \sqrt{c^2 - b^2} = \sqrt{1058 \times 512};$$

$$\therefore \log a = \frac{1}{2}\{\log 1058 + 9\log 2\} = 2{\cdot}8668778;$$

$$\therefore a = 736, \text{ and } \cos A = \frac{b}{c};$$

$$\therefore L\cos A = 10 + 2{\cdot}4361626 - 2{\cdot}8948697 = 9{\cdot}5412929;$$

$$\therefore A = 69^\circ.\ 38'.\ 56''{\cdot}3, \text{ and } B = 20^\circ.\ 21'.\ 3''{\cdot}7.$$

(8) $$a = \sqrt{c^2 - b^2} = \sqrt{1250 \times 32} = 200,$$

$$\cos A = \frac{b}{c};$$

$$\therefore L\cos A = 10 + 2{\cdot}7846173 - 2{\cdot}8068580 = 9{\cdot}9777593;$$

$$\therefore A = 18^\circ.\ 10'.\ 50'', \text{ and } B = 71^\circ.\ 49'.\ 10''.$$

(9) $$c = \sqrt{a^2 + b^2} = \sqrt{76176 + 243049} = 565,$$

$$\tan A = \frac{a}{b};$$

$$\therefore L\tan A = 10 + 2{\cdot}4409091 - 2{\cdot}6928469 = 9{\cdot}7480622;$$

$$\therefore A = 29^\circ.\ 14'.\ 30''{\cdot}3, \text{ and } B = 60^\circ.\ 45'.\ 29''{\cdot}7.$$

(10) $$c = \sqrt{a^2 + b^2} = \sqrt{156816 + 162409} = 565,$$

$$\tan A = \frac{a}{b};$$

$$\therefore L\tan A = 10 + 2{\cdot}5976952 - 2{\cdot}6053050 = 9{\cdot}9923902;$$

$$\therefore A = 44^\circ.\ 29'.\ 53'', \text{ and } B = 45^\circ.\ 30'.\ 7''.$$

EXAMPLES—LI. (p. 161).

(1) $\dfrac{\text{Height of steeple in feet}}{220} = \tan 46^\circ.\ 30'$, and if h be put for height of steeple,

$$\log h = \log 220 + L \tan 46^\circ.\ 30' - 10$$
$$= 2{\cdot}3424227 + {\cdot}0227500 = 2{\cdot}3651727\ ;$$
$$\therefore\ h = 231{\cdot}835 \text{ feet.}$$

(2) $\dfrac{BC}{AC} = \tan 25^\circ.\ 10'$, and if h be the height of the tower in feet,

$$\frac{h}{200} = \tan 25^\circ.\ 10'\ ;$$
$$\therefore\ \log h = \log 200 + L\,.\ \tan 25^\circ.\ 10' - 10$$
$$= \log 1000 - \log 5 + 9{\cdot}6719628 - 10$$
$$= 3 - {\cdot}6989700 + 9{\cdot}6719628 - 10$$
$$= 1{\cdot}9729928\ ;$$
$$\therefore\ h = 93{\cdot}97 \text{ feet.}$$

(3) $BC = 50$ feet ; $\angle BAC = 45^\circ$; $\angle BDC = 30^\circ$.

Then $AC = BC = 50$ feet.

(a) $AD = CD - AC$
$= BC\,.\ \cot 30^\circ - 50$
$= 50.(\cot 30^\circ - 1) = 50.(\sqrt{3} - 1)$
$= 50 \times {\cdot}7320508\ .\ .\ .$
$= 36{\cdot}6025\ .\ .\ .$ feet.

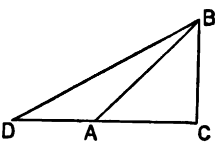

FIG. 24.

(β) $AB = AC\,.\ \sec 45^\circ = 50\,.\ \sqrt{2} = 50 \times 1{\cdot}4142\ldots = 70{\cdot}71\ldots$ feet.

(γ) $BD = BC\,.\ \operatorname{cosec} 30^\circ = 50 \times 2 = 100$ feet.

(4) If h be the measure of the height in feet,

$$\frac{h}{140} = \tan 54^\circ.\ 27'\ ;$$
$$\therefore\ h = 140 \times 1{\cdot}399364 = 195{\cdot}910960\ ;$$
$$\therefore \text{ height is 196 feet nearly.}$$

(5) Let PC be the hill.

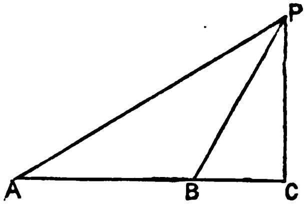

Fig. 25.

Then $\angle PAC = 32°. 14'$, and

$\angle PBC = 63°. 26'$.

Then $PC = BC \tan PBC$,

and $PC = AC . \tan PAC$.

$\therefore BC.\tan PBC = AC . \tan PAC$;

$\therefore BC \times 1{\cdot}998 = (500 + BC) \times {\cdot}63$,

whence $BC = 230$ nearly.

Hence $PC = 230 \times 1{\cdot}998 = 459{\cdot}54$

$= 460$ yards nearly.

(6) Let θ represent the sun's altitude.

$$\text{Then } \tan\theta = \frac{150}{75} = 2;$$

$$\therefore L \tan\theta = 10 + \log 2 = 10{\cdot}3010300.$$

$$\text{Hence } \theta = 63°. 26'. 6''.$$

(7) Let BC be the breadth of the river.

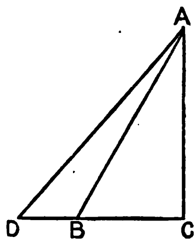

Fig. 26.

Then $AC = BC . \tan 60°$,

and $AC = CD . \tan 50°$.

$\therefore BC . \tan 60° = (40 + BC) \tan 50°$;

$\therefore BC \times \sqrt{3} = (40 + BC) \times 1{\cdot}19$;

$\therefore BC . (1{\cdot}73 - 1{\cdot}19) = 40 \times 1{\cdot}19$;

$\therefore {\cdot}54\, BC = 47{\cdot}6$,

and $\therefore BC = 88$ yards nearly.

(8) Let θ be the angle of inclination.

$$\text{Then } \sin\theta = \frac{60}{109} = {\cdot}55045.$$

$$\text{Hence } \theta = 33°. 23'. 55''{\cdot}7.$$

(9) Let θ be the angle of inclination.

$$\text{Then } \sin\theta = \frac{140}{221} = {\cdot}6306306;$$

$$\therefore \theta = 39°. 5'. 47''{\cdot}9.$$

(10) Let PC be the tower; $\angle PAC = 55°$; $\angle PBC = 48°$.

$$\text{Then } \frac{PA}{AB} = \frac{\sin 48°}{\sin BPA},$$

$$\text{or } \frac{PA}{30} = \frac{\sin 48°}{\sin 7°};$$

$$\therefore PA = 30 \times \frac{\sin 48°}{\sin 7°},$$

and $AC = PA \cdot \cos PAC = PA \cdot \sin 35°$.

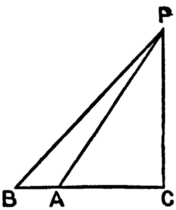

FIG. 27.

Hence if b be the breadth of the river in feet,

$$b = 30 \times \sin 35° \times \frac{\sin 48°}{\sin 7°}$$

$$\begin{aligned}\therefore \log b &= \log 30 + L\sin 35° + L\sin 48° - L\sin 7° - 10 \\ &= 1{\cdot}47712 + 9{\cdot}75859 + 9{\cdot}87107 - 9{\cdot}08589 - 10 \\ &= 2{\cdot}02089;\end{aligned}$$

$$\therefore b = 104{\cdot}93 \text{ feet.}$$

(11) Let AB be the height of the house, BD the length, C the place of observation.

Then ABC and CBD are right angles.

Then $BC = BD \cdot \cot BCD$,

and since $\cos BCD = \frac{1}{\sqrt{5}}$, $\cot BCD = \frac{1}{2}$;

$$\therefore BC = 150 \times \frac{1}{2} = 75 \text{ feet.}$$

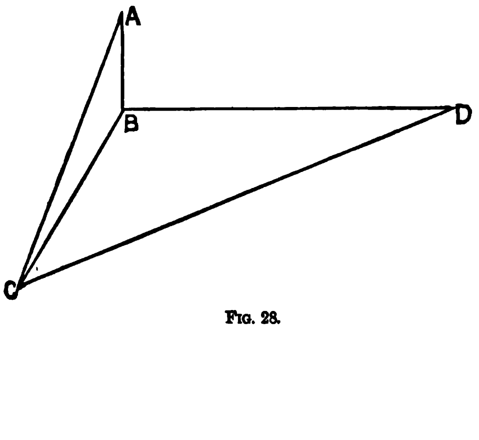

FIG. 28.

Again, $AB = BC \cdot \tan ACB$,

and since $\sin ACB = \dfrac{3}{\sqrt{34}}$, $\tan ACB = \dfrac{3}{5}$;

$$\therefore\ AB = 75 \times \frac{3}{5} = 45 \text{ feet.}$$

(12) Making the same construction as in Example (11),

$$BC = AB \cdot \cot ACB = 45 \times \frac{5}{3} = 75 \text{ feet,}$$

and $BD = BC \cdot \tan BCD = 75 \times 2 = 150$ feet.

EXAMPLES—LII. (p. 174).

(1) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{169 + 1600 - 1369}{1040} = \dfrac{5}{13}$;

$$\therefore\ \sin A = \frac{12}{13} = \cdot 9230769.$$

Hence $A = 67°.\ 22'.\ 48''\cdot 5$.

(2) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{841 + 14400 - 10201}{6960} = \dfrac{63}{87}$;

$$\therefore\ \sin A = \frac{60}{87} = \cdot 6896552.$$

Hence $A = 43°.\ 36'.\ 10''\cdot 1$.

(3) $s = \dfrac{1}{2}(37 + 13 + 30) = 40$;

$$\therefore\ \sin\frac{A}{2} = \sqrt{\frac{27 \times 10}{13 \times 30}} = \sqrt{\frac{9}{13}};$$

$$\therefore\ L\sin\frac{A}{2} = 10 + \frac{1}{2}\left\{\cdot 9542425 - 1\cdot 1139434\right\}$$

$$= 10 - \cdot 0798504 = 9\cdot 9201496.$$

Hence $A = 112°.\ 37'.\ 11''\cdot 5$.

(4) $s=\frac{1}{2}(409+241+600)=625$;

$$\therefore \sin A=\frac{2}{bc}\sqrt{s.(s-a)(s-b)(s-c)}$$
$$=\frac{2}{144600}\sqrt{625\times 216\times 384\times 25}$$
$$=\frac{2\times 36000}{144600}=\frac{360}{723};$$

$\therefore L\sin A=10+2\cdot 5563025-2\cdot 8591383=9\cdot 6971642.$

Hence $A=29^\circ.\ 51'.\ 46''\cdot 1.$

2.
$$\frac{\tan\frac{C-A}{2}}{\tan\frac{C+A}{2}}=\frac{c-a}{c+a};$$

$$\therefore \tan\frac{C-A}{2}=\frac{c-a}{c+a}.\cot\frac{B}{2}.$$

Now $c-a=1859$ and $c+a=13419$;

$$\therefore L\tan\frac{C-A}{2}=\log(c-a)-\log(c+a)+L\cot\frac{B}{2};$$

$$\therefore L\tan\frac{C-A}{2}=3\cdot 26928-4\cdot 12772+10\cdot 40312$$
$$=9\cdot 54468.$$

Hence $\frac{C-A}{2}=19^\circ.\ 18'.\ 50''.$

Also $\frac{C+A}{2}=68^\circ.\ 26'.\ 0''$;

$\therefore C=87^\circ.\ 44'.\ 50''$, and $A=49^\circ.\ 7'.\ 10''.$

3.
$$b=a.\frac{\sin B}{\sin A};$$

$$\therefore \log b=\log a+L\sin B-L\sin A$$
$$=1\cdot 7403627+9\cdot 9764927-9\cdot 8188779$$
$$=1\cdot 8979775;$$

$\therefore b=79\cdot 063.$

4. $$b=c\cdot\frac{\sin B}{\sin C};$$

$$\therefore \log b=\log c+L\sin B-L\sin C$$
$$=2\cdot1613680+9\cdot9982047-9\cdot8183919$$
$$=2\cdot3411808;$$
$$\therefore b=219\cdot37.$$

5. $$\sin A=\sin B\cdot\frac{a}{b};$$

$$\therefore L\sin A=L\sin B+\log a-\log b$$
$$=9\cdot7175280+2\cdot7537623-2\cdot5465269$$
$$=9\cdot9247634.$$

Hence *one* value of A is $57^\circ.\ 14'.\ 21''$.

And since a is greater than b, A is greater than B, and we may have the same given parts in a triangle where A is the supplement of $57^\circ.\ 14'.\ 21''$, or $122^\circ.\ 45'.\ 39''$.

6. $$\sin B=\frac{b}{c}\cdot\sin C=\frac{16}{8}\cdot\sin 30^\circ=\frac{2}{1}\times\frac{1}{2}=1;$$

$\therefore B=90^\circ$, and the triangle is not ambiguous.

7. In the equilateral triangle $a=b=c$;

$$\therefore \cos A=\frac{a^2+a^2-a^2}{2a^2}=\frac{a^2}{2a^2}=\frac{1}{2}.$$

8. Let $A=60^\circ$, $\frac{b}{c}=\frac{19}{1}$, and $\therefore \frac{b-c}{b+c}=\frac{18}{20}=\frac{9}{10}$.

$$\text{Now }\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cdot\cot\frac{A}{2}$$
$$=\frac{9}{10}\times\frac{\sqrt{3}}{1}=\frac{3^2\times3^{\frac{1}{2}}}{10}=\frac{3^{\frac{5}{2}}}{10};$$
$$\therefore L\tan\frac{B-C}{2}=10+\frac{5}{2}\log 3-\log 10$$
$$=10+1\cdot1928032-1$$
$$=10\cdot1928032;$$
$$\therefore \frac{B-C}{2}=57^\circ.\ 19'.\ 11'',$$
$$\text{and }\frac{B+C}{2}=60^\circ.\ 0'.\ 0''.$$
$$\therefore B=117^\circ.\ 19'.\ 11'',\text{ and }C=2^\circ.\ 40'.\ 49''.$$

9. Let a, b, c denote the sides in order of the given values.

$$\text{Then } \cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{6+(1+\sqrt{3})^2-4}{2(1+\sqrt{3}).\sqrt{6}} = \frac{6+2\sqrt{3}}{2\sqrt{6}+6\sqrt{2}} = \frac{1}{\sqrt{2}};$$

$$\therefore\ A = 45°.$$

$$\text{Again, } \sin B = \frac{b}{a}\cdot\sin A = \frac{\sqrt{6}}{2}\cdot\frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2};$$

$$\therefore\ B = 60°;$$

$$\text{and } \therefore\ C = 180° - (60°+45°) = 180° - 105° = 75°.$$

10. Construct a diagram, as in Art. 213, fig. 2, but with A and B interchanged, because B is here to be the *smaller* angle.

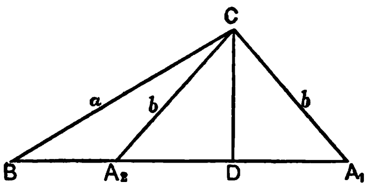

FIG. 29.

Let $c_1 = A_2B$, and $c_2 = A_1B$.

$$\text{Then } c_1 = BD - A_2D = a\cos B - b.\cos CA_2D,$$

$$\text{and } c_2 = BD + A_1D = a\cos B + b.\cos CA_2D;$$

$$\begin{aligned}\therefore\ c_1.c_2 &= a^2.\cos^2 B - b^2.\cos^2 CA_2D\\ &= a^2.\cos^2 B - b^2.\cos^2 A\\ &= a^2.(1-\sin^2 B) - b^2.(1-\sin^2 A)\\ &= a^2 - b^2;\\ \therefore\ c_1.c_2 + b^2 &= a^2.\end{aligned}$$

11. Let $A = 64°.12'$, and $\frac{b}{c} = \frac{9}{7}$.

$$\text{Then } \frac{b-c}{b+c} = \frac{9-7}{9+7} = \frac{2}{16} = \frac{1}{8}.$$

$$\begin{aligned}\text{And } \tan\frac{B-C}{2} &= \frac{b-c}{b+c}\cdot\cot\frac{A}{2}\\ &= \frac{1}{8}\cdot\cot 32°.6'.\end{aligned}$$

$$\therefore\ L\tan\frac{B-C}{2}=\log 1-\log 8+L\cot 32^\circ.\,6'$$
$$=0-3\log 2+L\tan 57^\circ.\,54'$$
$$=-\cdot 90309+10\cdot 2025255$$
$$=9\cdot 2994355.$$

$$\text{Hence } \frac{B-C}{2}=11^\circ.\,16'.\,10'',$$

$$\text{and } \frac{B+C}{2}=57^\circ.\,54'.\,0'';$$

$$\therefore\ B=69^\circ.\,10'.\,10'',\text{ and } C=46^\circ.\,37'.\,50''.$$

12. $$s=\frac{15}{2},\ s-a=\frac{7}{2},\ s-b=\frac{5}{2},\ s-c=\frac{3}{2}.$$

$$\therefore\ \cos\frac{B}{2}=\sqrt{\frac{15\times 5}{2.2.4.6}}=\sqrt{\frac{25}{2^5}};$$

$$\therefore\ L\cos\frac{B}{2}=10+\frac{1}{2}\left\{2\log 5-5\log 2\right\}$$
$$=10+\frac{1}{2}\left\{1\cdot 3979400-1\cdot 5051495\right\}$$
$$=9\cdot 9463953.$$

$$\text{Hence } \frac{B}{2}=27^\circ.\,53'.\,8'',\text{ and } B=55^\circ.\,46'.\,16''.$$

13. $$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cdot\cot\frac{C}{2}$$
$$=\frac{70-35}{70+35}\cdot\cot\frac{C}{2}$$
$$=\frac{1}{3}\cot.\,18^\circ.\,26'.\,6'';$$

$$\therefore\ L\tan\frac{A-B}{2}=\log 1-\log 3+L\cot 18^\circ.\,26'.\,6''$$
$$=0-\cdot 4771213+10\cdot 4771213$$
$$=10;$$

$$\therefore\ \frac{A-B}{2}=45^\circ,$$

$$\text{and } \frac{A+B}{2}=71^\circ.\,33'.\,54'';$$

$$\therefore\ A=116^\circ.\,33'.\,54'',\text{ and } B=26^\circ.\,33'.\,54''.$$

EXAMPLES—LIII. (p. 176).

(1) $$c=\sqrt{a^2+b^2}=\sqrt{16+9}=5,$$

$$\sin A=\frac{4}{5}=\cdot 8.$$

By the tables $\sin 53°.\ 7'=\cdot 7998593,$
$\sin 53°.\ 8'=\cdot 8000338.$

Hence $A=53°.\ 7'.\ 48''\cdot 4$, and $B=36°.\ 52'.\ 11''\cdot 6$.

$$a=\sqrt{c^2-b^2}=48,$$

$$\sin B=\frac{55}{73}=\cdot 7535068.$$

By the tables $\sin 48°.\ 53'=\cdot 7533721,$
$\sin 48°.\ 54'=\cdot 7535634.$

Hence $B=48°.\ 53'.\ 16''\cdot 5$, and $A=41°.\ 6'.\ 43''\cdot 5$.

(3) $$c=\sqrt{a^2+b^2}=353,$$

$$\sin A=\frac{272}{353}=\cdot 7705382.$$

By the tables $\sin 50°.\ 24'=\cdot 7705132$
$\sin 50°.\ 25'=\cdot 7706986.$

Hence $A=50°.\ 24'.\ 8''\cdot 1$, and $B=39°.\ 35'.\ 51''\cdot 9$.

(4) $$a=\sqrt{c^2-b^2}=40,$$

$$\sin A=\frac{40}{401}=\cdot 0997506.$$

By the tables $\sin 5°.\ 43'=\cdot 0996092,$
$\sin 5°.\ 44'=\cdot 0998986.$

Hence $A=5°.\ 43'.\ 29''\cdot 3$, and $B=84°.\ 16'.\ 30''\cdot 7$.

(5) $$B=79°.\ 7'.\ 9''\cdot 6.$$

By the tables $\sin 10°.\ 52'=\cdot 1885241,$
$\sin 10°.\ 53'=\cdot 1888098.$

Hence $\sin 10°.\ 52'.\ 50''\cdot 4=\cdot 1887639$;

$\therefore\ a=c\,.\,\sin A=445\times\cdot 1887639=84,$

and $b=\sqrt{c^2-a^2}=437.$

(6) $B = 43°.\ 0'.\ 10''{\cdot}3.$

By the tables $\sin 46°.\ 59' = {\cdot}7311553,$

$\sin 47°.\ 0' = {\cdot}7313537.$

Hence $\sin 46°.\ 59'.\ 49''{\cdot}7 = {\cdot}7313196$;

$\therefore a = c\,.\,\sin A = 629 \times {\cdot}7313196 = 460,$

and $b = \sqrt{c^2 - a^2} = 429.$

(7) $A = 38°.\ 34'.\ 48''{\cdot}3.$

By the tables $\sin 51°.\ 25' = {\cdot}7817019,$

$\sin 51°.\ 26' = {\cdot}7818833.$

Hence $\sin 51°.\ 25'.\ 11''{\cdot}7 = {\cdot}7817372$;

$\therefore b = c\,.\,\sin B = 449 \times {\cdot}7817372 = 351,$

and $a = \sqrt{c^2 - b^2} = 280.$

(8) $A = 31°.\ 2'.\ 53''{\cdot}6.$

By the tables $\sin 58°.\ 57' = {\cdot}8567175,$

$\sin 58°.\ 58' = {\cdot}8568675.$

Hence $\sin 58°.\ 57'.\ 6''{\cdot}4 = {\cdot}8567335$;

$\therefore b = c\,.\,\sin B = 349 \times {\cdot}8567335 = 299,$

and $a = \sqrt{c^2 - b^2} = 180.$

(9) $B = 23°.\ 57'.\ 8''.$

By the tables $\tan 23°.\ 57' = {\cdot}4441834,$

$\tan 23°.\ 58' = {\cdot}4445318.$

Hence $\tan 23°.\ 57'.8'' = {\cdot}4442365$;

$\therefore b = a\,.\,\tan B = 520 \times {\cdot}4442365 = 231,$

and $c = \sqrt{a^2 + b^2} = 569.$

(10) $B = 3°.\ 41'.\ 43''.$

By the tables $\tan 86°.\ 18' = 15{\cdot}463814,$

$\tan 86°.\ 19' = 15{\cdot}533981.$

Hence $\tan 86°.\ 18'.\ 17'' = 15{\cdot}483694$;

$\therefore a = b\,.\,\tan A = 31 \times 15{\cdot}483694 = 480,$

and $c = \sqrt{a^2 + b^2} = 481.$

EXAMPLES—LIV. (p. 177).

(1) $s=245$, $s-a=48$, $s-b=192$, $s-c=5$.

$$\text{Then } \tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s.(s-a)}};$$

$$\therefore L\tan\frac{A}{2}=10+\frac{1}{2}\left\{\log 192+\log 5-\log 245-\log 48\right\}$$

$$=10+\frac{1}{2}\left\{2{\cdot}2833012+{\cdot}6989700-2{\cdot}3891661-1{\cdot}6812412\right\}$$

$$=10+\frac{1}{2}\left\{2{\cdot}9822712-4{\cdot}0704073\right\}$$

$$=9{\cdot}4559320.$$

Hence $\frac{A}{2}=15°.\ 56'.\ 43''{\cdot}4$, and $\therefore\ A=31°.\ 53'.\ 26''{\cdot}8$.

By a similar method we may find $B=8°.\ 10'.\ 16''{\cdot}4$,

and $\therefore\ C=139°.\ 56'.\ 16''{\cdot}8$.

(2.) $s=605$, $s-a=96$, $s-b=384$, $s-c=125$.

$$L\tan\frac{A}{2}=10+\frac{1}{2}\left\{\log 384+\log 125-\log 605-\log 96\right\}$$

$$=10+\frac{1}{2}\left\{2{\cdot}5843312+2{\cdot}0969100-2{\cdot}7817554-1{\cdot}9822712\right\}$$

$$=10+\frac{1}{2}\left\{4{\cdot}6812412-4{\cdot}7640266\right\}$$

$$=9{\cdot}9586703.$$

Hence $\frac{A}{2}=42°.\ 16'.\ 25''{\cdot}25$, and $\therefore\ A=84°.\ 32'.\ 50''{\cdot}5$.

By a similar method we find $B=25°.\ 36'.\ 30''{\cdot}7$,

and $\therefore\ C=69°.\ 50'\ 38''{\cdot}8$.

(3) $s=680$, $s-a=147$, $s-b=363$, $s-c=170$.

$$L\tan\frac{A}{2}=10+\frac{1}{2}\left\{\log 363+\log 170-\log 680-\log 147\right\}$$
$$=10+\frac{1}{2}\left\{2{\cdot}5599066+2{\cdot}2304489-2{\cdot}8325089-2{\cdot}1673173\right\}$$
$$=10+\frac{1}{2}\left\{4{\cdot}7903555-4{\cdot}9998262\right\}$$
$$=9{\cdot}8952647.$$

Hence $\frac{A}{2}=38°.\ 9'.\ 26''$, and $\therefore A=76°.\ 18'.\ 52.''$

By a similar method we find $B=35°.\ 18'.\ 0''{\cdot}9$,

and $\therefore C=68°.\ 23'.\ 7''{\cdot}1$.

(4) $s=808$, $s-a=243$, $s-b=363$, $s-c=202$.

$$L\tan\frac{A}{2}=10+\frac{1}{2}\left\{\log 363+\log 202-\log 808-\log 243\right\}$$
$$=10+\frac{1}{2}\left\{2{\cdot}5599066+2{\cdot}3053514-2{\cdot}9074114-2{\cdot}3856063\right\}$$
$$=10+\frac{1}{2}\left\{4{\cdot}8652580-5{\cdot}2930177\right\}$$
$$=9{\cdot}7861202.$$

Hence $\frac{A}{2}=31°.\ 25'.\ 46''{\cdot}45$, and $\therefore A=62°.\ 51'.\ 32''{\cdot}9$.

By a similar method we find $B=44°.\ 29'.\ 53''$,

and $\therefore C=72°.\ 38'.\ 34''{\cdot}1$.

(5) $s=416$, $s-a=7$, $s-b=175$, $s-c=234$.

$$L\tan\frac{A}{2}=10+\frac{1}{2}\left\{\log 175+\log 234-\log 416-\log 7\right\}$$
$$=10+\frac{1}{2}\left\{2{\cdot}2430380+2{\cdot}3692159-2{\cdot}6190933-{\cdot}8450980\right\}$$
$$=10+\frac{1}{2}\left\{4{\cdot}6122539-3{\cdot}4641913\right\}$$
$$=10{\cdot}5740313.$$

Hence $\frac{A}{2}=75°.\ 4'.\ 7''$, and $\therefore A=150°.\ 8'.\ 14''$.

By a similar method we can find $B=17°.\ 3'.\ 41''{\cdot}5$,

and $\therefore C=12°.\ 48'.\ 4''{\cdot}5$.

(6) $$B=180^\circ-(A+C)=11^\circ.\ 25'.\ 16''\cdot 3,$$
$$a=b.\frac{\sin A}{\sin B}=\frac{29\times \cdot 6896550}{\cdot 1980199}=101,$$
$$c=b\cdot\frac{\sin C}{\sin B}=\frac{29\times \cdot 8193229}{\cdot 1980199}=120.$$

(7) $$B=180^\circ-(A+C)=39^\circ.\ 18'.\ 27''\cdot 5,$$
$$a=b\cdot\frac{\sin A}{\sin B}=\frac{149\times \cdot 9395972}{\cdot 6338400}=221,$$
$$c=b\cdot\frac{\sin C}{\sin B}=\frac{149\times \cdot 9438490}{\cdot 6338400}=222.$$

(8) $$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2},$$
$$\tan\frac{A-B}{2}=\frac{72}{130}\cdot\cot 16^\circ.\ 5'.\ 26''\cdot 9,$$
$$L\tan\frac{A-B}{2}=\log 72-\log 130+L\cot 16^\circ.\ 5'.\ 26''\cdot 9$$
$$=1\cdot 8573325-2\cdot 1139434+10\cdot 5399616$$
$$=10\cdot 2833507.$$
$$\text{Hence }\frac{A-B}{2}=62^\circ.\ 29'.\ 16''\cdot 8,$$
$$\text{and }\frac{A+B}{2}=73^\circ.\ 54'.\ 33''\cdot 1\ ;$$
$$\therefore\ A=136^\circ.\ 23'.\ 49''\cdot 9,\text{ and } B=11^\circ.\ 25'.\ 16''\cdot 3.$$
$$\text{Also, } c=\frac{a.\sin C}{\sin A}=\frac{101\times \cdot 5326047}{\cdot 6896550}=78.$$

(9) $$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2},$$
$$\tan\frac{A-B}{2}=\frac{360}{442}\cot 48^\circ.\ 28'.\ 40''\cdot 05,$$
$$L\tan\frac{A-B}{2}=\log 360-\log 442+L\cot 48^\circ.\ 28'.\ 40''\cdot 05$$
$$=2\cdot 5563025-2\cdot 6454223+9\cdot 9471473$$
$$=9\cdot 8580275.$$

$$\text{Hence } \frac{A-B}{2} = 35°.\ 47'.\ 50''\cdot65,$$

$$\text{and } \frac{A+B}{2} = 41°.\ 31'.\ 19''\cdot95\ ;$$

$$\therefore\ A = 77°.\ 19'.\ 10''\cdot6, \text{ and } B = 5°.\ 43'.\ 29''\cdot2.$$

$$\text{Also, } c = \frac{a\ .\ \sin C}{\sin A} = \frac{401 \times \cdot9926403}{\cdot9756097} = 408.$$

(10)
$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cdot\cot\frac{C}{2},$$

$$\tan\frac{A-B}{2} = \frac{72}{370}\cot 15°.\ 20'.\ 17''\cdot5,$$

$$L\tan\frac{A-B}{2} = \log 72 - \log 370 + L\cot 15°.\ 20'.\ 17''\cdot5$$
$$= 1\cdot8573325 - 2\cdot5682017 + 10\cdot5617669$$
$$= 9\cdot8508977.$$

$$\text{Hence } \frac{A-B}{2} = 35°.\ 21'.\ 15'',$$

$$\text{and } \frac{A+B}{2} = 74°.\ 39'.\ 42''\cdot5\ ;$$

$$\therefore\ A = 110°.\ 0'.\ 57''\cdot5, \text{ and } B = 39°.\ 18'.\ 27''\cdot5.$$

$$\text{Also, } c = \frac{a\ .\ \sin C}{\sin A} = \frac{221 \times \cdot5101885}{\cdot9395972} = 120.$$

(11)
$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cdot\cot\frac{C}{2},$$

$$\tan\frac{A-B}{2} = \frac{48}{170}\cdot\cot 33°.\ 29'.\ 42''\cdot7,$$

$$L\tan\frac{A-B}{2} = \log 48 - \log 170 + L\cot 33°.\ 29'.\ 42''\cdot7$$
$$= 1\cdot6812412 - 2\cdot2304489 + 10\cdot1792962$$
$$= 9\cdot6300885.$$

$$\text{Hence } \frac{A-B}{2} = 23°.\ 6'.\ 57''\cdot3,$$

$$\text{and } \frac{A+B}{2} = 56°.\ 29'.\ 42''\cdot7\ ;$$

$$\therefore\ A = 79°.\ 36'.\ 40'', \text{ and } B = 33°.\ 23'.\ 54''\cdot6.$$

$$\text{Also, } c = \frac{a\ .\ \sin C}{\sin A} = \frac{109 \times \cdot9204413}{\cdot9836064} = 102.$$

(12) $$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cdot\cot\frac{C}{2},$$

$$\tan\frac{A-B}{2}=\frac{362}{528}\cdot\cot 43°.\,57'.\,30'',$$

$$L\tan\frac{A-B}{2}=\log 362-\log 528+L\cot 43°.\,57'.\,30''$$

$$=2\cdot5587086-2\cdot7226339+10\cdot0157949$$

$$=9\cdot8518696.$$

$$\text{Hence } \frac{A-B}{2}=35°.\,24'.\,46'',$$

$$\text{and } \frac{A+B}{2}=46°.\,2'.\,30'';$$

$$\therefore A=81°.\,27'.\,16'', \text{ and } B=10°.\,37'.\,44''.$$

$$\text{Also, } c=\frac{b\,.\,\sin C}{\sin B}=\frac{83\times\cdot999390}{\cdot1844460}=450.$$

(13) $$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cdot\cot\frac{C}{2},$$

$$\tan\frac{A-B}{2}=\frac{120}{338}\cdot\cot 65°.\,42'.\,22'',$$

$$L\tan\frac{A-B}{2}=\log 120-\log 338+L\cot 65°.\,42'.\,22''$$

$$=2\cdot0791812-2\cdot5289167+9\cdot6545508$$

$$=9\cdot2048153.$$

$$\text{Hence } \frac{A-B}{2}=9°.\,6'.\,16''\cdot6,$$

$$\text{and } \frac{A+B}{2}=24°.\,17'.\,38'';$$

$$\therefore A=33°.\,23'.\,54''\cdot6, \text{ and } B=15°.\,11'.\,21''\cdot4.$$

$$\text{Also, } c=\frac{b\,.\,\sin C}{\sin B}=\frac{109\times\cdot7499700}{\cdot2620086}=312.$$

(14) $$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2},$$

$$\tan\frac{A-B}{2}=\frac{72}{410}\cdot\cot 52^\circ.\ 1'.\ 55''\cdot 5,$$

$$L\tan\frac{A-B}{2}=\log 72-\log 410+L\cot 52^\circ.\ 1'.\ 55''\cdot 5$$
$$=1\cdot 8573325-2\cdot 6127839+9\cdot 8923085$$
$$=9\cdot 1368571.$$
$$\therefore\ \frac{A-B}{2}=7^\circ.\ 48'.\ 12'',$$
$$\text{and }\frac{A+B}{2}=37^\circ.\ 58'.\ 4''\cdot 5\ ;$$
$$\therefore\ A=45^\circ.\ 46'.\ 16''\cdot 5,\text{ and }B=30^\circ.\ 9'.\ 52''\cdot 5.$$
$$\text{Also, }c=\frac{b\sin C}{\sin B}=\frac{169\times\cdot 9900242}{\cdot 5024855}=332\cdot 97.$$

(15) $$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2},$$

$$\tan\frac{A-B}{2}=\frac{72}{410}\cot 7^\circ.\ 41'.\ 18''\cdot 5,$$

$$L\tan\frac{A-B}{2}=\log 72-\log 410+L\cot 7^\circ.\ 41'.\ 18''\cdot 5$$
$$=1\cdot 8573325-2\cdot 6127839+10\cdot 8696637$$
$$=10\cdot 1142123.$$
$$\text{Hence }\frac{A-B}{2}=52^\circ.\ 26'.\ 54''\cdot 1,$$
$$\text{and }\frac{A+B}{2}=82^\circ.\ 18'.\ 41''\cdot 5\ ;$$
$$\therefore\ A=134^\circ.\ 45'.\ 36''\cdot 6,\text{ and }B=29^\circ.\ 51'.\ 46''\cdot 4.$$
$$\text{Also, }c=\frac{b\,.\sin C}{\sin B}=\frac{169\times\cdot 2651681}{\cdot 4982927}=90.$$

(16) $$\sin B = \frac{b \cdot \sin A}{a} = \frac{37 \times \sin 18°.\ 55'.\ 28''\cdot 7}{13};$$

$$\therefore L \sin B = \log 37 + L \sin 18°.\ 55'.\ 28''\cdot 7 - \log 13$$
$$= 1\cdot 5682017 + 9\cdot 5109783 - 1\cdot 1139434$$
$$= 9\cdot 9652366;$$

$\therefore B = 67°.\ 22'.\ 48''\cdot 1$, or its supplement $112°.\ 37'.\ 11''\cdot 9$.

(17) $$\sin B = \frac{b \cdot \sin A}{a} = \frac{565 \times \sin 44°.\ 29'.\ 53''}{445};$$

$$\therefore L \sin B = \log 565 + L \sin 44°.\ 29'.\ 53'' - \log 445$$
$$= 2\cdot 7520484 + 9\cdot 8456468 - 2\cdot 6483600$$
$$= 9\cdot 9493352;$$

$\therefore B = 62°.\ 51'.\ 32''\cdot 9$, or its supplement $117°.\ 8'.\ 27''\cdot 1$.

18) $$\sin B = \frac{b \cdot \sin A}{a} = \frac{836\cdot 4 \times \sin 14°.\ 24'.\ 25''}{212\cdot 5};$$

$$\therefore L \sin B = \log 836\cdot 4 + L \sin 14°.\ 24'.\ 25'' - \log 212\cdot 5$$
$$= 2\cdot 9224140 + 9\cdot 3958630 - 2\cdot 3273589$$
$$= 9\cdot 9909181;$$

$\therefore B = 78°.\ 19'.\ 24''$, or its supplement $101°.\ 40'.\ 36''$.

(19) $$\sin B = \frac{b \cdot \sin A}{a} = \frac{564\cdot 8 \times \sin 40°.\ 32'.\ 16''}{379\cdot 5};$$

$$\therefore L \sin B = \log 564\cdot 8 + L \sin 40°.\ 32'.\ 16'' - \log 379\cdot 5$$
$$= 2\cdot 7518947 + 9\cdot 8128794 - 2\cdot 5792118$$
$$= 9\cdot 9855623;$$

$\therefore B = 75°.\ 18'.\ 28''\cdot 2$, or its supplement $104°.\ 41'.\ 31''\cdot 8$.

(20) $$\sin B = \frac{b \cdot \sin A}{a} = \frac{8032\cdot 29 \times \sin 71°.\ 3'.\ 34''\cdot 7}{9459\cdot 31};$$

$$\therefore L \sin B = \log 8032\cdot 29 + L \sin 71°.\ 3'.\ 34''\cdot 7 - \log 9459\cdot 31$$
$$= 3\cdot 9048393 + 9\cdot 9758256 - 3\cdot 9758594$$
$$= 9\cdot 9048055;$$

$$\therefore B = 53°.\ 26'.\ 0''\cdot 6.$$

EXAMPLES—LV. (p. 181).

(1) Let QP be the hill; $\angle QBP = 60°$; $\angle QAP = 45°$.

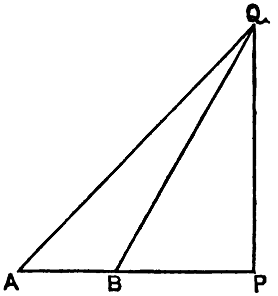

FIG. 30.

$$\begin{aligned} \text{Then } QP &= BP \,.\, \tan 60° \\ &= (AP - 100) \tan 60° \\ &= (QP - 100) \,.\, \sqrt{3}\,; \end{aligned}$$

$$\therefore QP = \frac{100\sqrt{3}}{\sqrt{3}-1} = \frac{100\sqrt{3}(\sqrt{3}+1)}{3-1} = 150 + 50\sqrt{3} = 236{\cdot}602 \ldots \text{ feet.}$$

(2) Let F be the fort; S_1 and S_2 the ships.

Then $\angle FS_1S_2 = 35°.\ 14'$, and $\angle FS_2S_1 = 42°.\ 12'$,

and $\angle S_1FS_2 = 180° - 77°.\ 26'$

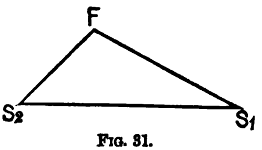

FIG. 31.

$$\text{and } FS_1 = S_1S_2 \cdot \frac{\sin FS_2S_1}{\sin S_1FS_2}$$

$$= 1760 \cdot \frac{\sin 42°.\ 12'}{\sin 77°.\ 26'}$$

$$= 1760 \times \frac{{\cdot}671}{{\cdot}976} = 1210 \text{ yards,}$$

$$\text{and } FS_2 = 1760 \times \frac{\sin 35°.\ 14'}{\sin 77°.\ 26'} = 1760 \times \frac{{\cdot}577}{{\cdot}976} = 1040{\cdot}5 \text{ yards.}$$

(3) With a construction similar to that in Example (2),

$$FS_1 = 880 \cdot \frac{\sin 85°.\ 15'}{\sin 11°} = 880 \times \frac{{\cdot}9965}{{\cdot}1908} = 4596 \text{ yards nearly,}$$

$$FS_2 = 880 \cdot \frac{\sin 83°.\ 45'}{\sin 11°} = 880 \times \frac{{\cdot}9940}{{\cdot}1908} = 4584{\cdot}48 \text{ yards.}$$

(4) Let AB be the flagstaff; BP the tower; Q the place of observation.

Then $\tan BQA = \tan(AQP - BQP)$

$$= \frac{\tan AQP - \tan BQP}{1 + \tan AQP \cdot \tan BQP}$$

$$= \frac{2{\cdot}05 - 2}{1 + 2{\cdot}05 \times 2} = \frac{{\cdot}05}{5{\cdot}1} = \frac{1}{102};$$

$\therefore L \tan BQA = 10 + \log 1 - \log 102$

$= 10 - 2{\cdot}0086002$

$= 7{\cdot}9913998;$

$\therefore BQA = 33'.\ 42''.$

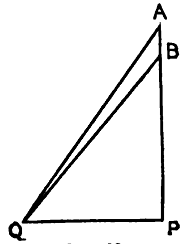

FIG. 32.

(5) Let A be the top of the steeple; D the top of the tower.

$\angle APB = 60°$ and $\angle DPB = 45°$.

Then $BA = PB \cdot \tan 60°$,

and $BD = PB \cdot \tan 45°$;

$\therefore BA : BD = \tan 60° : \tan 45°$

$= \sqrt{3} : 1.$

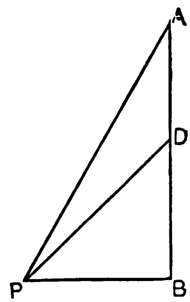

FIG. 33.

(6) Let PC be the river, CB the column, BA the statue $CD = 6$ feet; and let x = breadth of river in feet.

Then $\tan APB = \tan DPC = \dfrac{6}{x}$,

$\tan APC = \dfrac{AC}{PC} = \dfrac{230}{x}$,

$\tan BPC = \dfrac{200}{x}$.

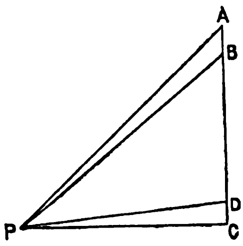

FIG 34.

$$\text{Now } \tan BPC = \tan(APC - APB);$$

$$\therefore \frac{200}{x} = \frac{\frac{230}{x} - \frac{6}{x}}{1 + \frac{230}{x} \cdot \frac{6}{x}};$$

$$\therefore \frac{200}{x} = \frac{224x}{x^2 + 1380}, \text{ or } 24x^2 = 276000, \text{ or } x^2 = 11500;$$

$$\therefore x = 107\cdot2 \ldots \text{ feet.}$$

(7) Let A be the top of the pole ; B the top of the mound.

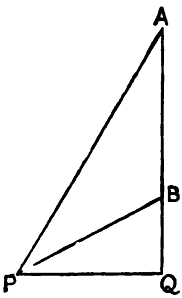

Fig. 35.

$$\angle APQ = 60^\circ; \ \angle BPQ = 30^\circ.$$

Then $AQ = PQ \cdot \tan 60^\circ$,

$BQ = PQ \cdot \tan 30^\circ$;

$$\therefore AQ : BQ = \tan 60^\circ : \tan 30^\circ$$

$$= \sqrt{3} : \frac{1}{\sqrt{3}}$$

$$= 3 : 1;$$

$$\therefore AB = 2BQ.$$

(8) Let A be the top of the flagstaff ; B the top of the tower.

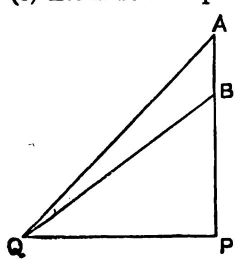

Fig. 36.

Then $\angle BQP = 90^\circ - \angle AQP$.

$$\text{Now } AB = AP - BP$$

$$= a(\tan AQP - \tan BQP)$$

$$= a.(\cot a - \tan a) = a \cdot \frac{\cos^2 a - \sin^2 a}{\cos a \,.\, \sin a}$$

$$= 2a \cdot \frac{\cos 2a}{\sin 2a} = 2a \,.\, \cot 2a.$$

(9) Let T be the place of the second observation.

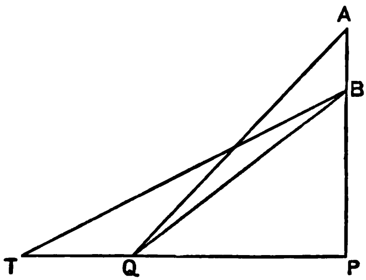

FIG. 37.

$$\text{Then } a = QP$$
$$= PT - TQ$$
$$= BP \,.\, \cot\frac{a}{2} - c$$
$$= a \tan a \,.\, \cot\frac{a}{2} - c;$$

$$\therefore\ c = a\left(\frac{2\tan\frac{x}{2}}{1-\tan^2\frac{a}{2}}\cot\frac{a}{2} - 1\right) = a\left(\frac{2}{1-\tan^2\frac{a}{2}} - 1\right) = a\cdot\frac{1+\tan^2\frac{a}{2}}{1-\tan^2\frac{a}{2}}$$

$$= a\cdot\frac{\cos^2\frac{a}{2}+\sin^2\frac{a}{2}}{\cos^2\frac{a}{2}-\sin^2\frac{a}{2}} = \frac{a}{\cos a};$$

$\therefore\ a = c\,.\,\cos a$, and putting this for a in the result of Example (8),

$$\text{length of flagstaff} = 2c\,.\,\cos a\,.\,\cot 2a = 2c\,.\,\cos a\cdot\frac{\cos 2a}{\sin 2a}$$

$$= 2c\cdot\frac{\cos 2a}{2\sin a} = c\,.\,\operatorname{cosec} a\,.\,\cos 2a.$$

(10) Let K be the kite; S_1 and S_2 the places of observation.

Draw KA perpendicular to S_1S_2.

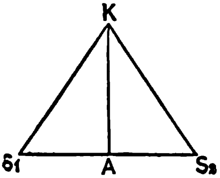

FIG. 38.

Then, since the angles at S_1 and S_2 are equal, KA bisects S_1S_2.

Then $KA = AS_1 . \tan KS_1A$

$$= \frac{a}{2} \cdot \tan\beta$$

$$= \frac{a}{2} \cdot \sin\beta . \sec\beta$$

$$= \frac{a}{2} \cdot \sin\alpha . \sec\beta, \text{ because } \alpha = \beta.$$

(11) Let AB be the smaller and PT the greater tower, and D the point midway between them.

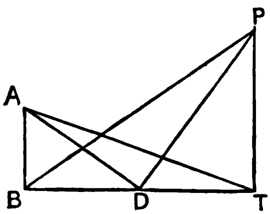

FIG. 39.

Join TA, BP, DA, DP.

Then $\angle PDT = \angle DAB$.

Let $PT = x$ and $AB = y$.

Then $\frac{x}{60} = \frac{60}{y}$, or $x = \frac{3600}{y}$.

Also, $\tan PBT = \tan 2ATB$

$$= \frac{2 \tan ATB}{1 - \tan^2 ATB},$$

and $\tan PBT = \frac{x}{120}$, and $\tan ATB = \frac{y}{120}$;

$$\therefore \frac{x}{120} = \frac{240y}{14400 - y^2};$$

$$\therefore \frac{3600}{120y} = \frac{240y}{14400 - y^2}.$$

Hence $y = 40$ feet, and $\therefore x = 90$ feet.

(12) Since $\angle ADB = \angle ACB$, a circle can be described about $ADCB$.

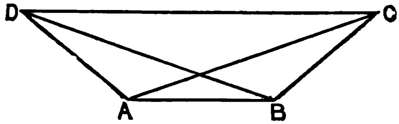

FIG. 40.

$$\therefore \angle ABD = \angle ACD = 19^\circ.15',$$

$$\text{and } \angle DAC = 180^\circ - (40^\circ.45' + 19^\circ.15') = 120^\circ.$$

$$\therefore AB = \frac{AD.\sin 30^\circ}{\sin 19^\circ.15'},$$

$$\text{and } AD = \frac{DC.\sin 19^\circ.15'}{\sin 120^\circ};$$

$$\therefore \frac{AB}{DC} = \frac{\sin 30^\circ}{\sin 120^\circ} = \frac{\sin 30^\circ}{\sin 60^\circ} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}.$$

(13) Let x be the length of the zigzag road in miles.

$$\text{Then } 5 : 12 = \frac{5}{3} : x;$$

$$\therefore 5x = 20, \text{ or } x = 4 \text{ miles.}$$

(14) S_1 and S_2 are the two positions of the ship, A and B the two objects.

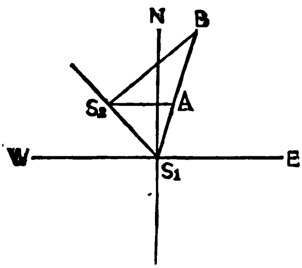

FIG. 41.

Then $\angle BS_1S_2 = 15^\circ + 45^\circ = 60^\circ$

$\angle BS_2S_1 = 90^\circ$(since N.W. is at right angles to N.E.)

$\angle S_2AS_1 = 180^\circ - (45^\circ + 60^\circ) = 75^\circ.$

Then $BS_1 = S_1S_2 . \sec . BS_1S_2$
$= 5 . \sec 60° = 10,$

$$\text{and } AS_1 = \frac{S_1S_2 . \sin AS_2S_1}{\sin S_2AS_1} = \frac{5 . \sin 45°}{\sin 75°} = \frac{5 \times \frac{1}{\sqrt{2}}}{\frac{\sqrt{3}+1}{2\sqrt{2}}} = \frac{10}{\sqrt{3}+1}.$$

$$\therefore AB = 10 - \frac{10}{\sqrt{3}+1} = \frac{10\sqrt{3}}{\sqrt{3}+1} = \frac{10\sqrt{3}(\sqrt{3}-1)}{3-1} = 5(3-\sqrt{3}).$$

(15) Let PQ be the tower.

Then AQP and PQB are right angles.

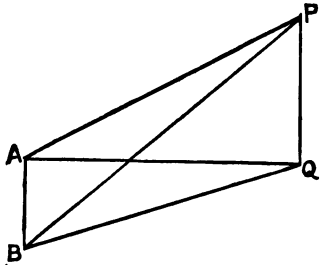

FIG. 42.

$\angle PAQ = 30°$, and $\angle PBQ = 18°$.

Then $AQ = PQ . \cot 30° = PQ \times \sqrt{3},$

$$BQ = PQ . \cot 18° = PQ \cdot \frac{\sqrt{(10+2\sqrt{5})}}{\sqrt{5}-1}. \text{ (See Example XXXVI. 6.)}$$

Now $BQ^2 - AQ^2 = a^2$;

$$\therefore PQ^2 \left\{ \frac{10+2\sqrt{5}}{6-2\sqrt{5}} - 3 \right\} = a^2;$$

$$\therefore PQ^2 \left\{ \frac{5+\sqrt{5}}{3-\sqrt{5}} - 3 \right\} = a^2;$$

$$\therefore PQ^2 \cdot \frac{4(\sqrt{5}-1)}{3-\sqrt{5}} = a^2;$$

$$\therefore PQ^2 \cdot \frac{4 . (2+2\sqrt{5})}{4} = a^2;$$

$$\therefore PQ = \frac{a}{\sqrt{(2+2\sqrt{5})}}.$$

(16) Let AB be the staff, C the centre of the ring in the vertical line ABC, D the extremity of the shadow; then if DE be drawn touching the ring in E, DE will be the direction of the sun, and CE is at right angles to DE.

Let $CE=r$, then $AB=AD=8r$, and $AC=9r$.

$$\therefore CD^2=AC^2+AD^2=145r^2,$$

$$\text{and } ED^2=CD^2-CE^2=144r^2;$$

$$\therefore ED=12r.$$

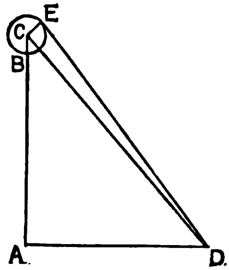

FIG. 43.

Hence $\tan ADC=\frac{9}{8}$, and $\tan CDE=\frac{1}{12}$;

$$\therefore \tan ADE=\frac{\frac{9}{8}+\frac{1}{12}}{1-\frac{9}{96}}=\frac{4}{3}.$$

$\therefore$ the sun's altitude $=\tan^{-1}\frac{4}{3}$.

(17) Draw DM, EN perpendicular to CB, and let AB, BC, CA be represented by c, a, b.

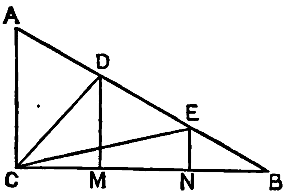

FIG. 44.

$$\text{Then } CD^2=CM^2+MD^2$$

$$=\frac{a^2}{9}+\frac{4b^2}{9} \qquad \text{(EUCLID, VI. 2, Ex. 1.)}$$

$$CE^2=CN^2+NE^2$$

$$=\frac{4a^2}{9}+\frac{b^2}{9}$$

$$DE^2=\frac{c^2}{9};$$

$$\therefore CD^2+CE^2+DE^2=\frac{5a^2}{9}+\frac{5b^2}{9}+\frac{a^2+b^2}{9}=\frac{2}{3}(a^2+b^2)=\frac{2}{3}c^2.$$

(18) Let P be the place of observation;

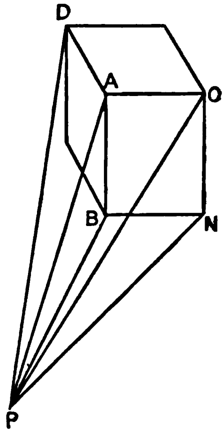

FIG. 45.

$BN=x$, $PB=a$.

Then $BA=a$, because $\angle APB=45°$,

$PN=a\,.\,\cot 30°=a\sqrt{3}$,

$\angle PBN=135°$.

Then $\cos PBN=\dfrac{PB^2+BN^2-PN^2}{2PB\,.\,BN}$,

or $\cos 135°=\dfrac{a^2+x^2-3a^2}{2ax}$,

$\therefore -\dfrac{1}{\sqrt{2}}=\dfrac{x^2-2a^2}{2ax}$,

$\therefore \dfrac{1}{2}=\dfrac{x^4-4a^2x^2+4a^4}{4a^2x^2}$.

Hence $x^4-6a^2x^2=-4a^4$, and $x^2-3a^2=\pm\sqrt{5}.a^2$,

and $\therefore\ x=a\sqrt{(3\pm\sqrt{5})}$.

(19) Let BA be the first tower; AC the moat; ED the other tower.

Draw EF parallel to DA. Let h=height of BA.

Then since $\angle BEA=\angle BCA=45°$, a circle can be described about $ABEC$, and since $\angle BAC=90°$, BC is the diameter of the circle, and therefore $\angle BEC=90°$.

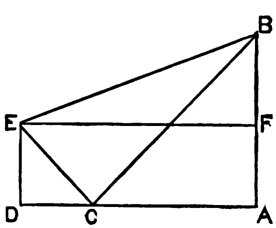

FIG. 46.

Then $CB^2=CA^2+BA^2=2h^2$,

and $CB^2=EC^2+EB^2$

$=a^2+c^2+EF^2+FB^2$

$=a^2+c^2+(c+h)^2+(h-a)^2$.

Hence

$2h^2=a^2+c^2+c^2+2ch+h^2+h^2-2ah+a^2$;

$\therefore\ h=\dfrac{a^2+c^2}{a-c}$.

(20) $$AC = AB \cdot \frac{\sin 15^\circ}{\sin 150^\circ} = 100 \cdot \frac{\sin 15^\circ}{\sin 30^\circ};$$

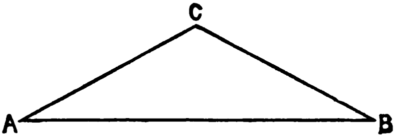

FIG. 47.

$$\therefore\ AC = 100 \times \frac{\sqrt{3}-1}{2\sqrt{2}} \div \frac{1}{2} = \frac{100(\sqrt{3}-1)}{\sqrt{2}}$$
$$= 50(\sqrt{6}-\sqrt{2}) = 51{\cdot}76\ \ldots\ \text{feet.}$$

(21) Since BC points to N.W. the $\angle\, ABC = 45^\circ$;

$\therefore\ \angle\, ACB = 45^\circ$, and $AC = AB = 10$ miles.

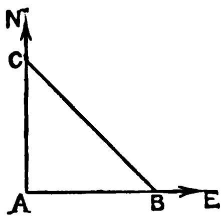

FIG. 48.

Also, $CB = \sqrt{AC^2 + AB^2} = \sqrt{200} = 14{\cdot}14\ \ldots$ miles.

(22) Let CA be a line from the end of the shadow in direction of the sun, AB the wall, BC the shadow.

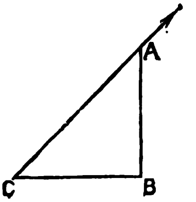

FIG. 49

Then $\tan ACB = \frac{AB}{BC} = \frac{18}{16} = \frac{9}{8}$.

$\therefore\ \tan ACB = 1{\cdot}125$;

or, $ACB = \tan^{-1} 1{\cdot}125$, which by the tables we find nearly equal to $48^\circ.\ 22'$.

(23) Let AB be the spire ; BP the tower ; Q the place of observation.

Then $\angle BQP = 30°$, and $\angle AQP = 32°$.

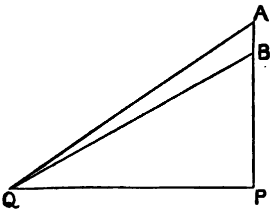

FIG. 50.

Now $AP = PQ \,.\, \tan 32° = 200 \times \cdot 6248694 = 124 \cdot 97398$

$BP = PQ \,.\, \tan 30° = 200 \times \cdot 5773503 = 115 \cdot 47006.$

$\therefore$ height of tower $= 115 \cdot 47$ yards nearly,

height of spire $= 9 \cdot 503$ yards nearly.

(24) $$\cos BAC = \frac{9 + 4 - \frac{324}{100}}{12} = \frac{61}{75} = \cdot 8133333.$$

Hence, by the tables, $\angle BAC = 35°.\,34'.\,32''$,

and $\therefore$ $\angle BAD = 144°.\,25'.\,28''$.

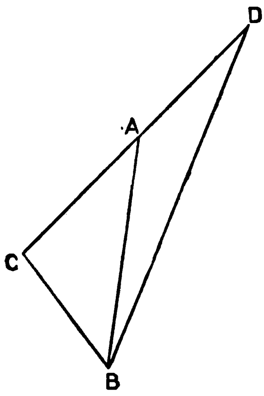

FIG. 51.

$$\text{Next, } BD = \frac{AB \,.\, \sin 144^\circ.\, 25'.\, 28''}{\sin 17^\circ.\, 47'.\, 20''}$$

$$= \frac{3 \times \cdot 5817759}{\cdot 3055106} = 5\cdot 71307 \ldots \text{ miles.}$$

(25) $\angle BAC = 17^\circ.\, 44'$,

$$AB = BC \cdot \frac{\sin 139^\circ.\, 58'}{\sin 17^\circ.\, 44'}.$$

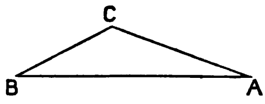

Fig. 52.

Hence, by the tables,

$$AB = \frac{840\cdot 5 \times \cdot 6432332}{\cdot 3045872} \text{yards} = 1775 \text{ yards nearly;}$$

$\therefore$ AB differs from a mile by about 15 yards.

(26) $\angle BCA = 180^\circ - (50^\circ.\, 20' + 110^\circ.\, 12') = 19^\circ.\, 28'$;

$$\therefore\ BC = AB \cdot \frac{\sin 50^\circ.\, 20'}{\sin 19^\circ.\, 28'}.$$

$$\therefore\ \log BC = \log 2700 + L \sin 50^\circ.\, 20' - L \sin 19^\circ.\, 28'$$

$$= 3\cdot 4313638 + 9\cdot 8863616 - 9\cdot 5227811$$

$$= 3\cdot 7949443.$$

Hence $BC = 6236\cdot 549$ feet.

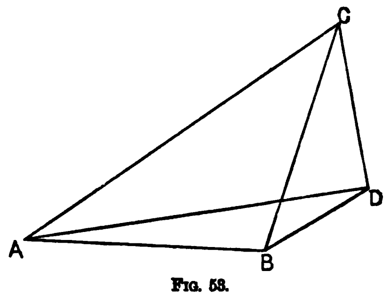

Fig. 53.

Next, if CD be the height of the mountain,

$$CD = BC \sin CBD,$$

$$= 6236\cdot 549 \times \sin 10^\circ.\, 7'$$

$$= 6236\cdot 549 \times \cdot 1756531$$

$$= 1095\cdot 47 \ldots \text{ feet.}$$

(27)

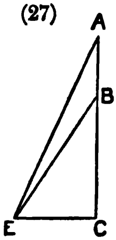

FIG. 54.

Let $EC = x$ feet.

Then $\tan AEB = \tan(AEC - BEC)$;

$$\therefore \tan 10^\circ = \frac{\tan AEC - \tan BEC}{1 + \tan AEC \cdot \tan BEC}.$$

$$\therefore \cdot 176327 = \frac{\frac{60}{x} - \frac{40}{x}}{1 + \frac{2400}{x^2}}.$$

$$\therefore \cdot 176327 = \frac{20x}{x^2 + 2400};$$

and solving this quadratic we get

$$x = 85{\cdot}28, \text{ or } 28{\cdot}14.$$

(28) Let CP be the height of the hill.

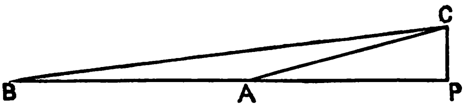

FIG. 55.

$$\text{Then } CA = AB \cdot \frac{\sin ABC}{\sin ACB}$$

$$= 1760 \times \frac{\sin 2^\circ.\ 45'}{\sin 9^\circ.\ 28'}$$

$$= \frac{1760 \times \cdot 0479781}{\cdot 1644738}$$

$$= 513{\cdot}4 \text{ nearly};$$

$$\therefore CP = 513{\cdot}4 \times \sin CAP$$

$$= 513{\cdot}4 \times \cdot 2116091 = 108{\cdot}64 \ldots \text{ yards.}$$

(29) Let AB be the tower, S the ship.

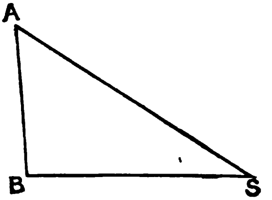

FIG. 56.

$$\text{Then } BS = AB \cdot \cot ASB$$
$$= 150 \times 1\cdot3613350$$
$$= 204\cdot2 \ldots \text{ feet.}$$

(30) $$\tan\frac{A-B}{2} = \frac{a-b}{a+b} \cdot \cot\frac{C}{2}$$
$$= \frac{3225\cdot77}{9541\cdot29} \cot 18^\circ.\ 43'.$$
$$L\tan\frac{A-B}{2} = 3\cdot5086333 - 3\cdot9795979 + 10\cdot4700495$$
$$= 9\cdot9990849.$$
$$\text{Hence } \frac{A-B}{2} = 44^\circ.\ 56'.\ 20'',$$
$$\text{and } \frac{A+B}{2} = 71^\circ.\ 17';$$
$$\therefore\ A = 116^\circ.\ 13'.\ 20'', \text{ and } B = 26^\circ.\ 20'.\ 40''.$$
$$\text{Also } c = \frac{b \cdot \sin C}{\sin B} = 3157\cdot76 \times \frac{\cdot6078379}{\cdot4437665} = 4325\cdot26.$$

(31) $\angle BS_1S_2 = 55^\circ$, and $\angle BS_2S_1 = 62^\circ.\ 30'$;

$\therefore \angle S_1BS_2 = 180^\circ - (55^\circ + 62^\circ.30') = 62^\circ.30'$;

$\therefore S_1S_2 = BS_1 = 1$ mile.

$$\text{Then } S_2B = \frac{S_1B \cdot \sin BS_1S_2}{\sin BS_2S_1}$$
$$= \frac{1 \times \sin 55^\circ}{\sin 62^\circ.30'} = \frac{\cdot8191520}{\cdot8870108}$$
$$= \cdot923497 \text{ miles.}$$

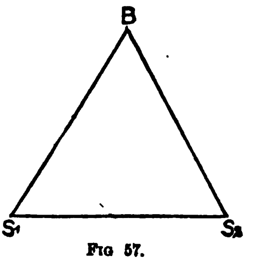

FIG. 57.

(32) From E, the lower window, draw EB perpendicular to the tower AB; from D, the upper window, draw DC perpendicular to the tower.

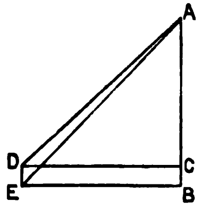

FIG. 58.

Then $\angle AEB = 45°$,

and $\angle ADC = 40°$,

and $DC = EB = AB$.

$\therefore DC = 20 + AC$,

$= 20 + DC \cdot \tan 40°$.

$$\therefore DC = \frac{20}{1 - \tan 40°} = \frac{20}{1 - \cdot 8390996}$$

$$= \frac{20}{\cdot 1609004} = 124 \cdot 3 \ldots \text{ feet.}$$

(33)

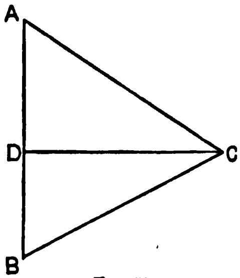

FIG. 59.

Let CD be the perpendicular breadth of the river.

Now $\angle ACB = 180° - (50° + 65°) = 65°$.

$\therefore AC = AB = 400$ yards.

Hence $CD = AC \cdot \sin 50°$

$= 400 \times \cdot 7660444 = 306 \cdot 4178$ yards.

(34) Let AB, AC be the lines of the railways, D the point at which the train travelling 30 miles an hour is in $2\frac{1}{2}$ hours.

The other train may then be at M or N, points on AB equidistant from D, and such that $MD = DN = 50$ miles.

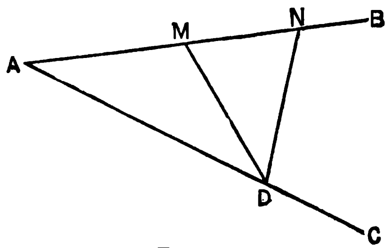

FIG. 60

Also, $AD = 75$ miles.

$$\text{Then } \sin AND = \frac{75 \,.\, \sin 35°.\, 20'}{50} = \frac{3}{2} \times \cdot5783323 = \cdot8674984.$$

$$\text{Hence } \angle AND = 60°.\, 10' \text{ nearly},$$

$$\therefore \angle ADN = 84°.\, 30',$$

$$\text{and } AN = \frac{50 \times \sin 84°.\, 30'}{\sin 35°.\, 20'} = \frac{50 \times \cdot9953962}{\cdot5783323} = \frac{49\cdot7698100}{\cdot5783323} \text{ miles.}$$

$$\therefore \text{rate of train} = \frac{49\cdot7698100}{\cdot5783323} \div 2\tfrac{1}{2} = 34\cdot42284 \ldots \text{ miles per hour.}$$

$$\text{Next, } \angle DMN = \angle AND = 60°.\, 10' \text{ nearly};$$

$$\therefore \angle AMD = 119°.\, 50' \text{ nearly};$$

$$\therefore \angle ADM = 24°.\, 50' \text{ nearly},$$

$$\text{and } AM = \frac{AD \,.\, \sin ADM}{\sin AMD} = \frac{75 \,.\, \sin 24°.\, 50'}{\sin 119°.\, 50'} = 75 \times \frac{\cdot4199801}{\cdot8674984} \text{ miles};$$

$$\therefore \text{rate of train} = 75 \times \frac{\cdot4199801}{\cdot8674984} \div 2\tfrac{1}{2} = 14\cdot524 \ldots \text{ miles per hour.}$$

(35) Let AB be the base of 600 yards; C the tree; CD a perpendicular on AB.

$$\text{Then } \angle ACB = 180° - (52°.\, 14' + 68°.\, 32')$$
$$= 59°.\, 14'.$$

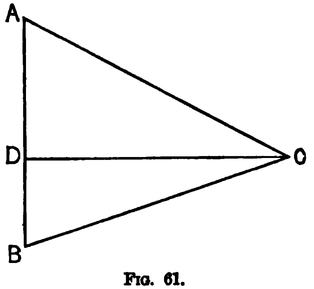

FIG. 61.

$$\text{Now } CD = AC \,.\, \sin CAD$$
$$= \frac{600 \,.\, \sin ABC}{\sin ACB} \,.\, \sin CAD$$
$$= \frac{600 \,.\, \sin 68°.\, 32' .\, \sin 52°.\, 14'}{\sin 59°.\, 14'}$$
$$= \frac{600 \times \cdot9306306 \times \cdot7905115}{\cdot8592576} = 513\cdot7045 \text{ yards.}$$

(36) Let AB be the tower ; C the first place of observation ; D the second place of observation.

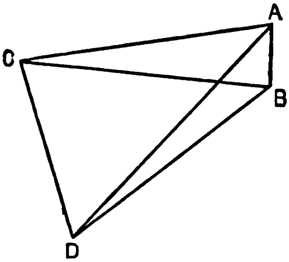

FIG. 62.

Then ACD and ABD are right angles.

Now $AC = AB \cdot \operatorname{cosec} ACB$

$= 100 \times \operatorname{cosec} 50° = 130 \cdot 54073.$

$AD = \sqrt{(300)^2 + (130 \cdot 54073)^2}$

$= \sqrt{107040 \cdot 127569}$

$= 327 \cdot 16 \ldots$

$\sin \angle ADB = \dfrac{AB}{AD} = \dfrac{100}{327 \cdot 16} = \cdot 3056608.$

Hence $\angle ADB = 17°.\ 47'.\ 50''$.

(37)

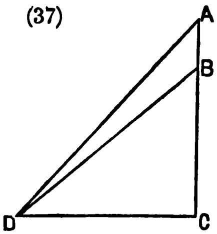

FIG. 63.

Let x = height of tower in yards ;

$\tan 48'.\ 20'' = \tan(ADC - BDC)$

$$= \frac{\dfrac{x+4}{100} - \dfrac{x}{100}}{1 + \dfrac{x \cdot (x+4)}{10000}};$$

$$\therefore \cdot 0140605 = \frac{400}{10000 + x^2 + 4x}.$$

Solving this quadratic we get $x = 134$ yards nearly.

(38) Let A be the object ; AB a vertical line meeting the horizontal plane through C in B ; D the point 300 yards up the hill.

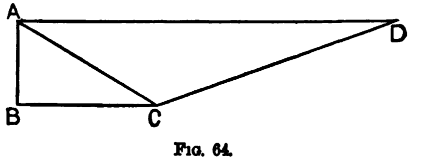

FIG. 64.

Then $\angle BCA = 29°.\ 12'.\ 40'' = \angle CAD.$

$\angle CDA = 16°.$

Then $CA = \dfrac{CD \cdot \sin 16°}{\sin 29°.\ 12'.\ 40''} = \dfrac{300 \times \cdot 2756374}{\cdot 4880290} = 169 \cdot 4392$ yards.

(39) At the end of three hours each engine has passed over 90 miles. Let AB, AC be the distances traversed.

Draw AD perpendicular to BC.

Then $\angle BAD = 25°.\ 10'$,

and $BD = AB \cdot \sin BAD$

$= 90 \times \cdot 4252528$.

$\therefore BC = 2 \times 90 \times \cdot 4252528 = 76 \cdot 5455 \ldots$ miles.

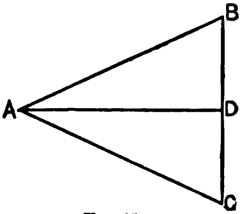

FIG. 65.

(40) Diagram as in Example (37); and x = height of tower in feet.

$$\tan ADB = \frac{6}{150} = \frac{1}{25};$$

$$\therefore \frac{1}{25} = \frac{\frac{x+30}{150} - \frac{x}{150}}{1 + \frac{x(x+30)}{22500}} = \frac{4500}{x^2 + 30x + 22500}.$$

Solving this quadratic $x = 285$ feet nearly.

EXAMPLES—LVI. (p. 199).

1. $\text{Area} = \frac{1}{2}\left\{10 \times 12 \times \sin 60°\right\}$ square inches

$= \left(60 \times \frac{\sqrt{3}}{2}\right)$ square inches $= 30\sqrt{3}$ square inches.

2. $\text{Area} = \frac{1}{2}\left\{40 \times 60 \times \sin 30°\right\}$ square feet

$= \left(1200 \times \frac{1}{2}\right)$ square feet $= 600$ square feet.

3. $\text{Area} = \frac{1}{2}\left\{4 \times 3\frac{3}{4}\right\}$ square feet $= 7\frac{1}{2}$ square feet.

4. $\text{Area} = \sqrt{s \cdot (s-a) \cdot (s-b) \cdot (s-c)} = \sqrt{8 \times 3 \times 2 \times 3} = 4 \times 3$

$= 12$ square inches.

5. $\text{Area} = \sqrt{1017 \times 392 \times 512 \times 113} = \sqrt{9 \times 113 \times 8 \times 49 \times 8 \times 64 \times 113}$

$= (3 \times 113 \times 8 \times 7 \times 8) = 151872.$

6. Area $=\sqrt{544 \times 135 \times 375 \times 34}=\sqrt{17 \times 32 \times 15 \times 9 \times 125 \times 3 \times 17 \times 2}$
$=17 \times 8 \times 9 \times 25=30600.$

7. Area $=\sqrt{585 \times 8 \times 512 \times 65}=\sqrt{5 \times 13 \times 9 \times 8 \times 64 \times 8 \times 13 \times 5}$
$=5 \times 13 \times 3 \times 8 \times 8=12480.$

8.
$$s\,.\,(s-c)=\frac{a+b+c}{2}\cdot\frac{a+b-c}{2}$$
$$=\frac{(a+b)^2-c^2}{4}$$
$$=\frac{(a+b)^2-(a^2+b^2)}{4}$$
$$=\frac{2ab}{4}$$
$$=\frac{ab}{2}=\text{area of the triangle.}$$

9. Area $=\sqrt{\frac{146{\cdot}27}{2}\times\frac{41{\cdot}21}{2}\times\frac{48{\cdot}75}{2}\times\frac{56{\cdot}31}{2}}$

$\therefore$ log area $=\frac{1}{2}\left\{\log 146{\cdot}27+\log 41{\cdot}21+\log 48{\cdot}75+\log 56{\cdot}31-4\log 2\right\}$

$=\frac{1}{2}\left\{2{\cdot}1651553+1{\cdot}6150026+1{\cdot}6879746\right.$
$\left.+1{\cdot}7505855-1{\cdot}2041200\right\}$

$=3{\cdot}0072990$;

$\therefore$ area $=1016{\cdot}9487.$

10. Let a, b, c be in descending arithmetical progression;
then $a+c=2b$.

Thus the perimeter is $3b$, and the side of an equilateral triangle of equal perimeter is b.

$$\text{Then } \sqrt{s\,.\,(s-a)(s-b)(s-c)}=\frac{3}{5}\cdot\frac{1}{2}\cdot b^2.\sin 60^\circ,$$

$$\text{or } \frac{1}{4}\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}=\frac{3\sqrt{3}}{20}b^2,$$

$$\text{or}\sqrt{3b^2(b+c-a)(a+b-c)}=\frac{3\sqrt{3}}{5}b^2$$

$$\sqrt{(b+c-a)(a+b-c)}=\frac{3}{5}b$$

$$\sqrt{\frac{3c-a}{2}\cdot\frac{3a-c}{2}}=\frac{3}{10}(a+c)$$

$$\frac{10ac-3a^2-3c^2}{4}=\frac{9}{100}(a^2+c^2).$$

Solving this quadratic we get $\frac{a}{c}=\frac{7}{3}$ or $\frac{3}{7}$.

Hence the sides are proportional to 7, 5, 3.

Then $\cos A=\frac{b^2+c^2-a^2}{2bc}=-\frac{1}{2}$;

and $\therefore A=120°$.

11. Let AEB be the triangular part turned down.

Then area of $AEB=\frac{1}{2}AB.AE.$

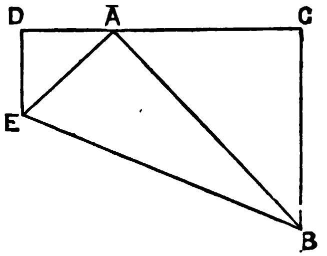

FIG. 66.

But $\frac{AE}{AD}=\frac{AB}{BC}$, by similar triangles AED, BAC;

$$\therefore \text{ area of } AEB=\frac{1}{2}AB\cdot\frac{AB.AD}{BC}$$

$$=\frac{1}{2}\cdot\frac{AB^2}{BC}\cdot(CD-AC)$$

$$=\frac{1}{2}\cdot\frac{AB^2}{BC}\cdot\left\{AB-\sqrt{(AB^2-BC^2)}\right\}$$

12. Area $=\dfrac{bc \,.\, \sin A}{2}=\dfrac{b\sin A \,.\, c\sin A}{2\sin A}=\dfrac{a\sin B \,.\, a\sin C}{2\sin A}=\dfrac{a^2\sin B \,.\, \sin C}{2\sin(B+C)}$.

13. $\sin\dfrac{A}{2}\cdot\sin\dfrac{B}{2}\cdot\sin\dfrac{C}{2}\left(\dfrac{a^2}{\sin A}+\dfrac{b^2}{\sin B}+\dfrac{c^2}{\sin C}\right)$

$$=\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{(s-a)(s-c)}{ac}}\cdot\sqrt{\frac{(s-a)(s-b)}{ab}}\cdot\left(\frac{a^2bc}{2S}+\frac{b^2ac}{2S}+\frac{c^2ba}{2S}\right)$$

$$=\frac{(s-a)(s-b)(s-c)}{abc}\cdot\frac{abc(a+b+c)}{2S}$$

$$=\frac{s.(s-a)(s-b)(s-c)}{S}=S.$$

14. $$R=\frac{abc}{4S} \text{ and } r=\frac{S}{s};$$

$$\therefore \frac{abc}{4S}=\frac{2S}{s};$$

$$\therefore abc=\frac{8S^2}{s}$$

$$=8(s-a)(s-b)(s-c)$$

$$=(b+c-a)(a+c-b)(a+b-c).$$

Squaring both sides,—

$$a^2b^2c^2=\{a+(b-c)\}\{a-(b-c)\}\times\{b+(a-c)\}\{b-(a-c)\}\times\{c+(a-b)\}\{c-(a-b)\}$$

$$=\{a^2-(b-c)^2\}\{b^2-(a-c)^2\}\{c^2-(a-b)^2\}.$$

Now this equality can only exist when $a=b=c$, for in any other case each factor on the right-hand side is less than the corresponding factor on the left-hand side.

15. $$\frac{b-c}{a}=\frac{\sin B-\sin C}{\sin A}=\frac{2\cos\frac{B+C}{2}\cdot\sin\frac{B-C}{2}}{2\sin\frac{A}{2}\cdot\cos\frac{A}{2}}=\frac{\sin\frac{B-C}{2}}{\cos\frac{A}{2}};$$

$$\therefore (b-c)\cos\frac{A}{2}=a\,.\,\sin\frac{B-C}{2}.$$

16. OA bisects $\angle A$, and FE at right angles;

$$\therefore \text{ area } FOE = FH \,.\, OH$$

$$= r\cos\frac{A}{2} \cdot r\sin\frac{A}{2}$$

$$= r^2 \cdot \frac{1}{2}\sin A$$

$$= \frac{S^2}{s^2} \cdot \frac{S}{bc}$$

$$= \frac{S^3}{s^2 \,.\, bc}.$$

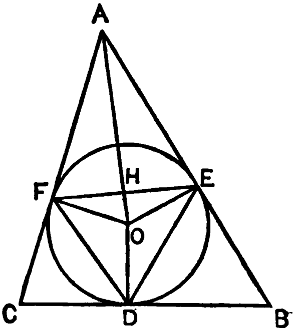

FIG. 67.

$\therefore$, by symmetry,

$$\text{area } FDE = \frac{S^3}{s^2}\left(\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab}\right)$$

$$= \frac{S^3 \,.\, 2s}{s^2 . abc}$$

$$= \frac{2}{abc} \cdot \frac{\{s \,.\, (s-a)(s-b)(s-c)\}^{\frac{3}{2}}}{s}$$

$$= \frac{2}{abc} \cdot s^{\frac{1}{2}} \left\{ (s-a)(s-b)(s-c) \right\}^{\frac{3}{2}}.$$

K

17. Area of quadrilateral

$$=\frac{1}{2}\left\{EA\,.\,AD+DA\,.\,AC+BA\,.\,AC+BA\,.\,AE\right\}\sin A$$

$$=\frac{1}{2}\left\{(EA+AC)\,.\,AD+(EA+AC)BA\right\}\sin A$$

$$=\frac{1}{2}\cdot EC\,.\,BD\,.\sin A$$

$$=\frac{1}{2}ab\,.\sin A.$$

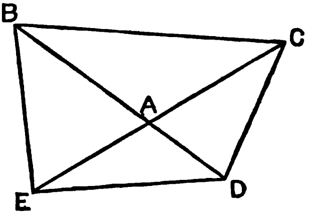

FIG. 68.

18. $$\frac{a^2-b^2}{2}\cdot\frac{\sin A\,.\sin B}{\sin(A-B)}=\frac{a^2\sin A\,.\sin B-b^2\sin A\,.\sin B}{2\sin(A-B)}$$

$$=\frac{ab\sin^2 A-ab\sin^2 B}{2\sin(A-B)}=\frac{ab.\sin(A+B).\sin(A-B)}{2\sin(A-B)}$$

$$=\frac{ab\sin(A+B)}{2}$$

$$=\frac{ab\,.\sin C}{2}=\text{area of triangle.}$$

19. $$R=\frac{a}{2\sin A}=\frac{a}{\sqrt{2}}$$

$$r_a=\frac{S}{s-a}=\frac{\frac{1}{2}ab}{\frac{2a+c}{2}-a}=\frac{ab}{c}=\frac{a^2}{a\sqrt{2}}=\frac{a}{\sqrt{2}};$$

$$\therefore\ R=r_a.$$

20. $\cot(B-A)+\cot 2\left(A+\frac{C}{2}\right)=\cot(B-A)+\cot(2A+C)$

$$=\frac{1+\cot B\,.\,\cot A}{\cot B-\cot A}+\frac{1-\cot 2A\,.\,\cot C}{\cot 2A+\cot C}$$

$$=\frac{1+1}{\tan A-\cot A}+\frac{1-0}{\cot 2A+0}$$

$$=\frac{2}{\tan A-\cot A}+\frac{2\tan A}{1-\tan^2 A}$$

$$=\frac{2\tan A}{\tan^2 A-1}+\frac{2\tan A}{1-\tan^2 A}$$

$$=0.$$

21. $\frac{2abc}{a+b+c}\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}$

$$=\frac{2abc}{a+b+c}\cdot\sqrt{\frac{s\,.\,(s-a)}{bc}}\cdot\sqrt{\frac{s\,.\,(s-b)}{ac}}\cdot\sqrt{\frac{s\,.\,(s-c)}{ab}}$$

$$=\frac{2abc}{a+b+c}\cdot\frac{s}{abc}\cdot\sqrt{s\,.\,(s-a)(s-b)(s-c)}$$

$$=\sqrt{s\,.\,(s-a)(s-b)(s-c)}$$

$=$ area of triangle.

22. $\frac{\sin 2A(2a+c)^2}{32\,.\,\cos^4\frac{A}{2}}=\frac{\sin 2A\,.\,(2s)^2}{32\,.\,\frac{s^2\,.\,(s-a)^2}{b^2c^2}}$

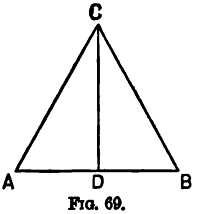

FIG. 69.

$$=\frac{\sin 2A\,.\,b^2c^2}{8\,.\,(s-a)^2}$$

$$=\frac{\sin 2A\,.\,b^2c^2}{8\left(\frac{c}{2}\right)^2}$$

$$=\frac{b^2\,.\sin 2A}{2}=b^2\,.\,\sin A\,.\,\cos A=b^2\,.\,\sin A\,.\,\frac{c}{2b}$$

$$=\frac{1}{2}bc\,.\,\sin A$$

$=$ area of triangle.

$\therefore$ area $\times\,32\cos^4\frac{A}{2}=\sin 2A\,.\,(2a+c)^2.$

23.

$$AD = b \,.\, \sin C, \;\therefore,\; AD \,.\, b = b^2 \,.\, \sin C.$$

$$AD = c \,.\, \sin B, \;\therefore,\; AD \,.\, c = c^2 \,.\, \sin B.$$

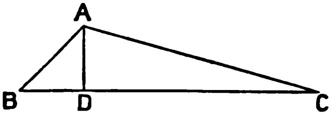

FIG. 70.

$$\therefore\; AD(b+c) = b^2 \,.\, \sin C + c^2 \,.\, \sin B\,;$$

$$\therefore\; AD = \frac{b^2 \sin C + c^2 \sin B}{b+c}.$$

24. (1)

$$BD = r \,.\, \cot\frac{B}{2}$$

$$CD = r \,.\, \cot\frac{C}{2}\,;$$

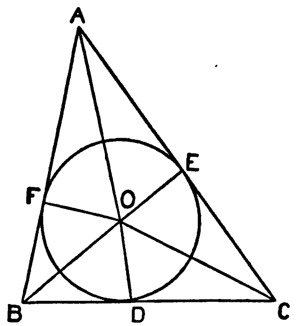

FIG. 71.

$$\therefore\; r \,.\left(\cot\frac{B}{2} + \cot\frac{C}{2}\right) = BD + CD = a.$$

$$\therefore\; r = \frac{a}{\cot\frac{B}{2} + \cot\frac{C}{2}}.$$

(2) From the preceding Example—

$$r=\frac{a\sin\frac{B}{2}\cdot\sin\frac{C}{2}}{\sin\left(\frac{B+C}{2}\right)}$$

$$=\frac{2R\,.\sin A\,.\sin\frac{B}{2}\cdot\sin\frac{C}{2}}{\cos\frac{A}{2}}, \text{ by Art. 221.}$$

(3) Let O be the centre of the escribed circle touching BC and the other sides produced, as in diagram to Art. 223.

Then $BD=OD\,.\cot OBD=r_1\,.\tan\frac{B}{2}$,

and $CD=OD\,.\cot OCD=r_1\,.\tan\frac{C}{2}$.

$\therefore\ BD+CD=r_1\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)$;

$$\therefore r_1=\frac{a}{\tan\frac{B}{2}+\tan\frac{C}{2}}.$$

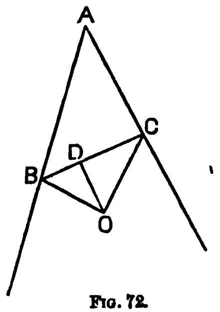

FIG. 72.

(4) By the preceding Example—

$$r_1=\frac{a\,.\cos\frac{B}{2}\cdot\cos\frac{C}{2}}{\sin\frac{B+C}{2}}$$

$$=\frac{2R\,.\sin A\,.\cos\frac{B}{2}\cdot\cos\frac{C}{2}}{\cos\frac{A}{2}}$$

$$=4R\,.\sin\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}.$$

(5) $$r_1=4R.\sin\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2},$$

$$r_2=4R.\cos\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\cos\frac{C}{2},$$

$$r_3=4R.\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\sin\frac{C}{2};$$

$$\therefore\ r_1+r_2+r_3=4R.\cos\frac{A}{2}\cdot\left(\sin\frac{B}{2}\cdot\cos\frac{C}{2}+\cos\frac{B}{2}\cdot\sin\frac{C}{2}\right)$$
$$+4R.\sin\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}$$

$$=4R.\cos\frac{A}{2}\cdot\cos\frac{A}{2}+4R.\sin\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}$$

$$=2R.(\cos A+1)+R.(\cos B+\cos C-\cos A+1)$$

Ex. XLVIII. 12.

$$=3R+R(\cos A+\cos B+\cos C).$$

(6) $$R+r=R+\frac{2R\sin A.\sin\frac{B}{2}\cdot\sin\frac{C}{2}}{\cos\frac{A}{2}}\ \text{by (2)}$$

$$=R+4R.\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}$$

$$=R.\left(1+4\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}\right)$$

$$=R(\cos A+\cos B+\cos C).$$ Ex. XLVIII. 8.

25. Let r be the radius of the circle.

Then area of inscribed polygon of $2n$ sides $=nr^2.\sin\frac{\pi}{n}$,

area of inscribed polygon of n sides $=\frac{nr^2}{2}\cdot\sin\frac{2\pi}{n}$;

area of circumscribed polygon of n sides $=nr^2.\tan\frac{\pi}{n}$.

$$\text{And}\left(\frac{nr^2}{2}\cdot\sin\frac{2\pi}{n}\right)\times\left(nr^2.\tan\frac{\pi}{n}\right)$$

$$\frac{n^2.r^4}{2}\cdot 2\sin\frac{\pi}{n}\cdot\cos\frac{\pi}{n}\cdot\frac{\sin\frac{\pi}{n}}{\cos\frac{\pi}{n}}$$

$$=n^2r^4.\sin^2\frac{\pi}{n}$$

$$=\left(nr^2.\sin\frac{\pi}{n}\right)^2$$

26. Let O, M be the centres of the inscribed and escribed circles.

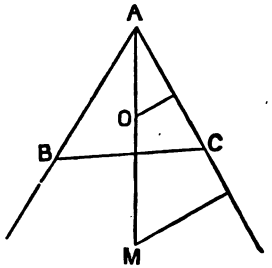

FIG. 73.

Then $MO=MA-OA$

$$=r_1\operatorname{cosec}\frac{A}{2}-r.\operatorname{cosec}\frac{A}{2}$$

$$=(r_1-r)\operatorname{cosec}\frac{A}{2}$$

$$=\left\{4R.\sin\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}-4R.\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}\right\}\operatorname{cosec}\frac{A}{2}$$

(By Ex. 24.)

$$=4R\left\{\cos\frac{B}{2}\cdot\cos\frac{C}{2}-\sin\frac{B}{2}\cdot\sin\frac{C}{2}\right\}$$

$$=4R.\sin\frac{A}{2},$$

and similarly for the other escribed circles.

(27) Let DEF be the triangle so formed.

Then since $\dfrac{BD}{CD}=\dfrac{c}{b}$,

$$\frac{BD}{BC}=\frac{c}{b+c}, \text{ or, } BD=\frac{ac}{b+c}.$$

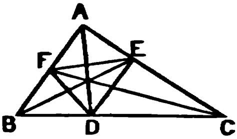

Fig. 74.

So also, $CD=\dfrac{ab}{b+c}$, and similarly for the segments of the other sides.

$$\text{Then area } CDE=\frac{1}{2}\cdot\frac{ab}{b+c}\cdot\frac{ab}{a+c}\cdot\sin C=\frac{S\,.\,ab}{(a+c)(b+c)}.$$

Similar expressions may be obtained for the areas of BFD, AFE.

$$\therefore \text{area of } DEF=S\left\{1-\frac{ab}{(a+c)(b+c)}-\frac{bc}{(b+a)(c+a)}-\frac{ca}{(c+b)(a+b)}\right\}$$

$$=\frac{2abc\,.\,S}{(a+b)(b+c)(c+a)}=2S\,.\,\frac{a}{b+c}\cdot\frac{b}{c+a}\cdot\frac{c}{a+b}.$$

$$\text{Now, } \frac{a}{b+c}=\frac{\sin A}{\sin B+\sin C}=\frac{\sin\frac{A}{2}}{\cos\frac{B-C}{2}}$$

$$\frac{b}{c+a}=\frac{\sin\frac{B}{2}}{\cos\frac{C-A}{2}}, \text{ and } \frac{c}{a+b}=\frac{\sin\frac{C}{2}}{\cos\frac{A-B}{2}}$$

$$\therefore \frac{\text{area}DEF}{\text{area}ABC}=\frac{2\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}}{\cos\frac{B-C}{2}\cdot\cos\frac{C-A}{2}\cdot\cos\frac{A-B}{2}}.$$

28. $r_1r_2+r_2r_3+r_3r_1=\dfrac{S^2}{(s-a)(s-b)}+\dfrac{S^2}{(s-b)(s-c)}+\dfrac{S^2}{(s-c)(s-a)}$

$$=s\,.\,(s-c)+s\,.\,(s-a)+s\,.\,(s-b)$$

$$=s\,.\,\{3s-(a+b+c)\}$$

$$=s^2.$$

29.
$$\frac{\sin BAD}{\sin ADB}=\frac{BD}{AB}.$$

$$\frac{\sin ABC}{\sin ACB}=\frac{AC}{AB}.$$

$\therefore$ since $\sin ADB=\sin ACB$,

$$\frac{\sin BAD}{\sin ABC}=\frac{BD}{AC};$$

$$\therefore AC\sin A=BD.\sin B.$$

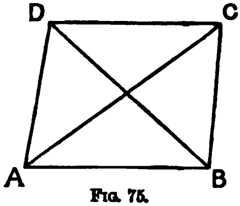

FIG. 75.

30. Let O, P be the centres of the inscribed and one of the escribed circles.

Then OB and PB bisect the interior and exterior angles at B; and $\therefore OBP$ is a right angle.

Hence $OBPC$ is a quadrilateral round which a circle may be described.

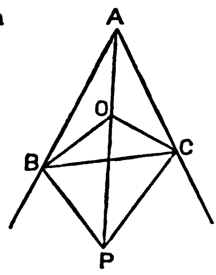

FIG. 76

$$\text{Then } OP=OB.\sec BOP$$
$$=OB.\sec BCP$$
$$=OB.\operatorname{cosec}\frac{C}{2}.$$

$$\text{And } OB=\frac{c.\sin\frac{A}{2}}{\sin AOB}=\frac{c.\sin\frac{A}{2}}{\cos\frac{C}{2}};$$

$$\therefore OP=\frac{c.\sin\frac{A}{2}}{\sin\frac{C}{2}\cdot\cos\frac{C}{2}}=\frac{2c.\sin\frac{A}{2}}{\sin C}=\frac{2a.\sin\frac{A}{2}}{\sin A}=\frac{a}{\cos\frac{A}{2}}.$$

$$\text{Similarly } OP=\frac{b}{\cos\frac{B}{2}}=\frac{c}{\cos\frac{C}{2}}.$$

31. $r \cdot \cos\frac{A}{2} \cdot \operatorname{cosec}\frac{B}{2} \cdot \operatorname{cosec}\frac{C}{2} = \dfrac{r\cos\frac{A}{2}}{\sin\frac{B}{2}\cdot\sin\frac{C}{2}}$,

$$= r \cdot \frac{\sqrt{\frac{s.(s-a)}{bc}}}{\sqrt{\frac{(s-a).(s-c)}{ac}} \cdot \sqrt{\frac{(s-b)(s-a)}{ab}}}$$

$$= r \cdot \frac{a\sqrt{s}}{\sqrt{(s-a)(s-b)(s-c)}}$$

$$= \frac{S}{s} \cdot \frac{a\sqrt{s}}{\sqrt{(s-a)(s-b)(s-c)}}$$

$$= a.$$

32. $\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}$

$$= \sqrt{\frac{(s-c)(s-b)}{s.(s-a)}} + \sqrt{\frac{(s-a)(s-c)}{s.(s-b)}} + \sqrt{\frac{(s-b)(s-a)}{s.(s-c)}}$$

$$= \frac{(s-c)(s-b)}{S} + \frac{(s-a)(s-c)}{S} + \frac{(s-b)(s-a)}{S}$$

$$= \frac{1}{4S} \cdot \left\{ (a+b-c).(a+c-b) + (b+c-a).(a+b-c) + (a+c-b).(b+c-a) \right\}$$

$$= \frac{1}{4S}\left\{ 2ab + 2ac + 2bc - a^2 - b^2 - c^2 \right\}$$

$$= \frac{1}{S} \cdot \left\{ ab + ac + bc - \frac{a^2+b^2+c^2+2ab+2ac+2bc}{4} \right\}$$

$$= \frac{1}{S}\left\{ ab + ac + bc - s^2 \right\}$$

$$= \frac{ab+ac+bc}{S} - \frac{s^2}{S}$$

$$= \frac{4R}{abc} \cdot (ab+ac+bc) - \frac{s}{r}$$

$$= 4R.\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - \frac{s}{r}.$$

33. $PA \cdot BC = BA \cdot PC + AC \cdot BP$ (EUCLID, VI. D.)

$$\text{and } \frac{\sin A}{BC} = \frac{\sin C}{BA} = \frac{\sin B}{AC}.$$

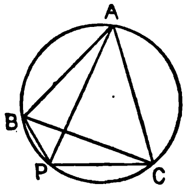

FIG. 77.

$$\therefore PA \cdot \sin A = PC \cdot \sin C + PB \cdot \sin B.$$

34. Each of the angles at $O = 120°$.

Let OA, OB, OC be represented by d_1, d_2, d_3.

$c^2 = d_1^2 + d_2^2 - 2d_1d_2 \cdot \cos 120°$;

$$\therefore c = \sqrt{d_1^2 + d_2^2 + d_1d_2}.$$

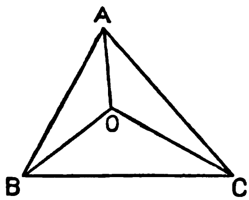

FIG. 78.

Similarly for a and b.

Also, area $= \left(\dfrac{d_1d_2}{2} + \dfrac{d_1d_3}{2} + \dfrac{d_2d_3}{2}\right)\sin 120°$

$$= \frac{\sqrt{3}}{4} \cdot (d_1d_2 + d_1d_3 + d_2d_3).$$

35. Let $OA=a$, $OB=b$, $OC=c$; $\angle OBA=\theta$, and let x be the side of the square $ABCD$.

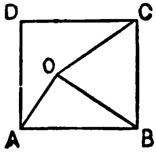

FIG. 79.

Then $\angle OBC=90°-\theta$,

and $a^2=x^2+b^2-2bx\cos\theta$;

$c^2=x^2+b^2-2bx\sin\theta$;

$\therefore\ 2bx\cos\theta=x^2+b^2-a^2$,

$2bx\sin\theta=x^2+b^2-c^2$.

Squaring and adding these equations,

$$4b^2x^2=x^4+2(b^2-a^2)x^2+(b^2-a^2)^2+x^4+2(b^2-c^2)x^2+(b^2-c^2)^2;$$

$$\therefore\ 2x^4-2(a^2+c^2)x^2+(a^2+c^2)^2=2(a^2b^2+a^2c^2+b^2c^2-b^4),$$

$$\text{and } x=\sqrt{\frac{1}{2}\left\{a^2+c^2\pm\sqrt{4(a^2b^2+a^2c^2+b^2c^2-b^4)-(a^2+c^2)^2}\right.}$$

(Gaskin's *Solutions of Trigonometrical Examples.*)

36. Let ABC be any triangle described about a circle.

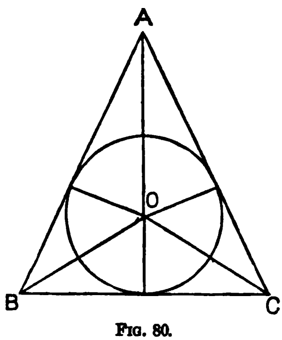

FIG. 80.

Then area of ABC=area of AOB+area of BOC+area of AOC.

$$\therefore\ \text{area of } ABC=\frac{1}{2}\cdot rc+\frac{1}{2}ra+\frac{1}{2}rb.$$

$$=\frac{r}{2}(a+b+c);$$

$\therefore$ since r is constant,

area of $ABC\propto(a+b+c)$.

37.
$$\alpha = AD = c \,.\, \sin B = b \,.\, \sin C,$$
$$\beta = BE = c \,.\, \sin A,$$
$$\gamma = CF = b \,.\, \sin A\,;$$
$$\therefore\ \frac{a^2}{\beta\gamma} = \frac{bc \,.\, \sin B \,.\, \sin C}{bc \,.\, \sin A \,.\, \sin A} = \frac{bc}{a^2}.$$

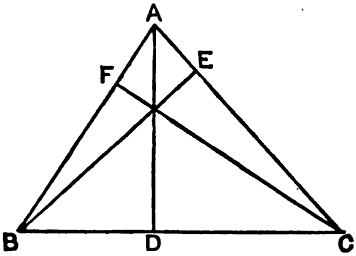

FIG. 81.

$$\text{Similarly } \frac{\beta^2}{\alpha\gamma} = \frac{ac}{b^2}\,;\ \text{and } \frac{\gamma^2}{\alpha\beta} = \frac{ab}{c^2}\,;$$
$$\therefore\ \frac{a^2}{\beta\gamma} + \frac{\beta^2}{\alpha\gamma} + \frac{\gamma^2}{\alpha\beta} = \frac{bc}{a^2} + \frac{ac}{b^2} + \frac{ab}{c^2}.$$

38. Let A be the observer on the sea-shore, O the earth's centre, BC the mountain whose height = 1284·8 yards = ·73 miles.

Then since C is just visible from A,

AC is a tangent at A.

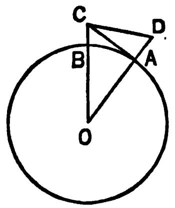

FIG. 82.

Join OA and produce it to D, making $AD=3$ miles; then $\angle DCA$ = angle of depression of C from $D = 2°.\,15'$.

$$\text{Then } AC = 3 \,.\, \cot 2°.\,15'$$
$$\log AC = \log 3 + L\cot 2°.\,15' - 10$$
$$= \cdot 4771213 + 11\cdot 4057168 - 10$$
$$= 1\cdot 8828381\,;$$
$$\therefore\ AC = 76\cdot 3551.$$

Let OA, the earth's radius, $=r$;

$$\therefore\ AC^2=BC(2r+BC)=\cdot73(2r+\cdot73),$$

and $\log(2r+\cdot73)=2\log AC-\log\cdot73=3\cdot9023533$;

$$\therefore\ 2r+\cdot73=7986\cdot4;$$

$$\therefore\ r=3992\cdot835 \text{ miles.}$$

(Gaskin's *Solutions of Trigonometrical Examples.*)

39. Let ABC be the triangle, $CO=b$, $BO=a$,

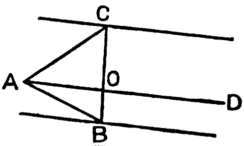

FIG. 83.

$\angle BAD=\theta$, and $\therefore\ \angle CAD=60^\circ-\theta$.

Let $AB=x$.

Then $x.\sin\theta=a$,

$x\sin(60^\circ-\theta)=b$;

$$\therefore\ \frac{\sin(60^\circ-\theta)}{\sin\theta}=\frac{b}{a}.$$

$$\therefore\ \frac{\sqrt{3}}{2}\cdot\cot\theta-\frac{1}{2}=\frac{b}{a}.$$

$$\therefore\ a.\cot\theta=\frac{3b+a}{\sqrt{3}}.$$

$$\therefore\ x=a.\operatorname{cosec}\theta=\sqrt{a^2+\frac{4b^2+4ab+a^2}{3}}=2\sqrt{\frac{a^2+ab+b^2}{3}}\quad\text{(Gaskin).}$$

40.

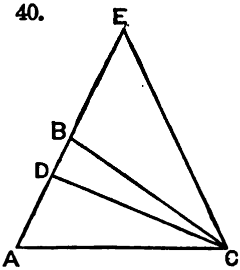

FIG. 84.

$$\frac{AD}{DB}=\frac{AC}{CB}=2;\ \therefore\ AD=2DB,$$

$$AB=AD+DB=3DB,$$

$$\frac{AE}{EB}=\frac{AC}{CB}=2;\ \therefore\ AE=2BE;$$

$$\therefore\ BE=AB=3DB;$$

$$\therefore\ DE=BE+DB=4DB.$$

Then, by EUCLID, VI. i.

$\triangle CBD : \triangle ACD : \triangle ABC : \triangle CDE$

$=DB : AD : AB : DE$

$=1\quad : 2\quad : 3\quad : 4.$ (Gaskin).

41. $$R\,.\sin A=\frac{a}{2},\text{ by Art. 221 };$$

$$\begin{aligned}\therefore\ Rr\,.(\sin A+\sin B+\sin C)&\\ &=r\,.\left(\frac{a+b+c}{2}\right)\\ &=r\,.\,s\\ &=\text{area of the triangle}\end{aligned}$$

42. The circles have the same radius because $R=\dfrac{b}{2\sin B}$.

In the example given, $\sin 50^\circ.\,15'=\cdot7688418$;

$$\therefore\ R=\frac{564}{1\cdot5376836}=366\cdot785.$$

43. Call the angles $x, \dfrac{x+y}{2}, \dfrac{x+2y}{2}, \dfrac{x+3y}{3}$.

$$\left.\begin{aligned}\text{Then } x+\frac{x+y}{2}+\frac{x+2y}{2}+\frac{x+3y}{3}&=2\pi\\ \text{and } x+\frac{x+2y}{2}&=\pi\end{aligned}\right\};$$

$$\therefore\ \left.\begin{aligned}14x+15y&=12\pi\\ 3x+2y&=2\pi\end{aligned}\right\}.$$

$$\text{Hence } x=\frac{6\pi}{17}\text{ and } y=\frac{8\pi}{17};$$

$$\therefore\ \text{the angles are } \frac{6\pi}{17}, \frac{7\pi}{17}, \frac{11\pi}{17}, \frac{10\pi}{17}$$

44. $$\frac{(1+\cot PCA)^2}{(1+\cot PCB)^2}=\frac{\left(1-\dfrac{CM}{PM}\right)^2}{\left(1+\dfrac{CM}{PM}\right)^2}$$

$$=\frac{(PM-CM)^2}{(PM+CM)^2}$$

$$=\frac{CP^2-2CN\,.\,PN}{CP^2+2CN\,.\,PN}$$

$$=\frac{CN\,.\,CD-2CN\,.\,PN}{CN\,.\,CD+2CN\,.\,PN}$$

$$=\frac{CO-PN}{CO+PN}=\frac{CB-CM}{AC+CM}=\frac{BM}{AM}=\frac{\cot PBA}{\cot PAB}.$$

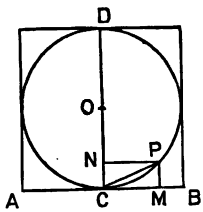

FIG. 85.

45. $r+r_a+r_b-r_c=\dfrac{S}{s}+\dfrac{S}{s-a}+\dfrac{S}{s-b}-\dfrac{S}{s-c}$

$$=\frac{S(2s-a)}{s\,.\,(s-a)}+\frac{S\,.\,(s-c-s+b)}{(s-b)(s-c)}$$

$$=\frac{S\,.\,(b+c)}{s\,.\,(s-a)}+\frac{S\,.\,(b-c)}{(s-b)(s-c)}$$

$$=S\,.\left\{\frac{b\,.\,\{(s-b)(s-c)+s\,.\,(s-a)\}+c\{(s-b)(s-c)-s\,.\,(s-a)\}}{S^2}\right\}$$

$$=\frac{b\{2s^2-s\,.\,(a+b+c)+bc\}+c\{-s(b+c-a)+bc\}}{S}$$

$$=\frac{b^2c-\dfrac{c}{2}(b+c+a)(b+c-a)+bc^2}{S}$$

$$=\frac{c}{2S}\cdot\left\{2b^2-(b+c)^2+a^2+2bc\right\}$$

$$=\frac{c}{2S}(b^2-c^2+a^2)=\frac{c}{2S}2ab\cos C=\frac{abc\,.\,\cos C}{S}=4R\,.\cos C.$$

46. Let C be the right angle; then, by Art. 223,

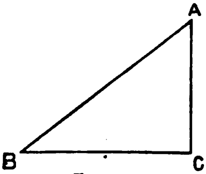

FIG. 86.

$$S=\frac{c+b-a}{2}\cdot r,\text{ and}$$

$$S=\frac{c+a-b}{2}\cdot r';$$

$$\therefore\ S^2=\frac{c^2-(a-b)^2}{4}\cdot rr'$$

$$=\frac{c^2-a^2+2ab-b^2}{4}\cdot rr'=\frac{ab}{2}rr'=S\,.\,rr';$$

$$\therefore\ rr'=S.$$

47.

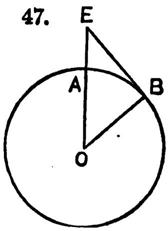

FIG. 87.

Using the notation of Art. 228,

$OB=4000$ miles,

$OE=OB\,.\sec.\,1^\circ.\,58'.\,10''$

$=4000\times1{\cdot}005910$

$=4002{\cdot}364$.

$\therefore\ AE=2{\cdot}36\ldots$ miles.

48. Using the notation of Art. 228,

$$\sec EOB = \frac{4001 \cdot 25}{4000} = 1 \cdot 0003125,$$

and, by the Tables, $\sec 1°.\, 26' = 1 \cdot 0003130$.

Hence dip of horizon $= 1°.\, 26'$ nearly.

49. Let A be the man's eye; B the lamp; C the centre of the earth.

Then $AD + DB = 52800$ feet.

And, if the radius of the earth be R feet,

$$AD^2 = 6(2R + 6),$$
$$BD^2 = 32(2R + 32).$$

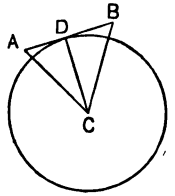

FIG. 88.

Hence, approximately, $\sqrt{12R} + \sqrt{64R} = 52800$,

or, $\sqrt{R} \,.\, (4 + \sqrt{3}) = 26400$, or, $\sqrt{R} \times 13 = 26400(4 - \sqrt{3})$.;

$$\therefore \; R = \frac{26400 \times 26400 \times (19 - 8\sqrt{3})}{13 \times 13 \times 1760 \times 3} \text{miles} = 4017 \cdot 79 \ldots \text{miles}.$$

50. In 72 minutes the ship travels 12 miles.

Then using the notation of Art. 228,

$$BE^2 = CE \,.\, EA,$$

$$144 = \left(CA + \frac{90}{5280}\right) \cdot \frac{90}{5280}$$

$$= CA \cdot \frac{90}{5280} \text{ nearly};$$

$$\therefore \; CA = \frac{144 \times 528}{9} = 16 \times 528 = 8448.$$

$\therefore$ radius $= 4224$ miles.

51. Using the notation of Art. 228,

$$\cos EOB = \frac{OB}{OE} = \frac{3956}{3959}.$$

$$\therefore\ L\cos EOB = 10 + \log 3956 - \log 3959$$
$$= 10 + 3{\cdot}5972563 - 3{\cdot}5975855$$
$$= 9{\cdot}9996708.$$

Whence, by the tables,

$$EOB = 2^\circ.\ 13'.\ 50''.$$

52. Let r be the radius of a section of the earth, made by a plane through its centre perpendicular to the line joining its centre with the sun's centre. Then if θ be the circular measure of the angle subtended by r at the sun's centre, and d be the distance between the two centres,

$$\frac{r}{d} = \tan\theta = \theta \text{ nearly, since } \theta \text{ is very small.}$$

$$\therefore\ \frac{r}{d} = \frac{8{\cdot}868}{57{\cdot}29577 \times 60 \times 60}.$$

$$\therefore\ d = \frac{57{\cdot}29577 \times 60 \times 60 \times 4000}{8{\cdot}868}$$

$$= \frac{206264772}{2{\cdot}217} = 93037786{\cdot}1\ \ldots \text{ miles}$$

53. Using the same notation as in Ex. 52,

$$\tan\theta = \frac{4000}{241118};$$

$$\therefore\ L\tan\theta = 10 + \log 4000 - \log 241118$$
$$= 10 + 3{\cdot}6020600 - 5{\cdot}3822296$$
$$= 8{\cdot}2198304.$$

Hence, by the tables,

$$\theta = 57'.\ 1''{\cdot}5 = \text{nearly}.$$

54. Let A, B be the two points; then AB is a tangent at its middle point D to the earth's surface.

$$AD = DE \text{ nearly} = 4 \text{ miles},$$

$$AE = 10 \text{ feet} = \frac{10}{5280} \text{miles}.$$

Let C be the earth's centre, and $CD = r$.

Then $AE(2r + AE) = AD^2$.

$\therefore$, approximately, $AE \cdot 2r = AD^2$;

$$\therefore r = \frac{16 \times 5280}{10 \times 2} = 4224 \text{ miles}.$$

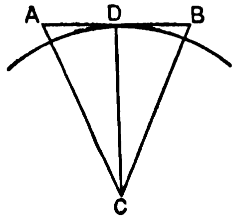

FIG. 89.

55. The limit of deviation is the angle subtended by the radius of the target at a point 600 feet distant, and if this angle be denoted by θ

$$\tan\theta = \frac{2}{600};$$

$$\therefore \theta = \tan^{-1} \cdot 00\dot{3}.$$

56. Regard the moon M as the base of a cone of which E, the eye of the observer, is the vertex. Then S, the shilling, will intercept all the rays of light from M to E, when it is so near to S that lines from E to the circumference of S do not, when produced, fall within the circumference of M.

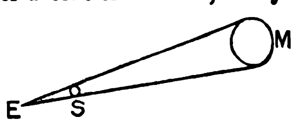

FIG. 90.

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